Principles of Interferometry

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IMPRS Black Board Lectures 2014

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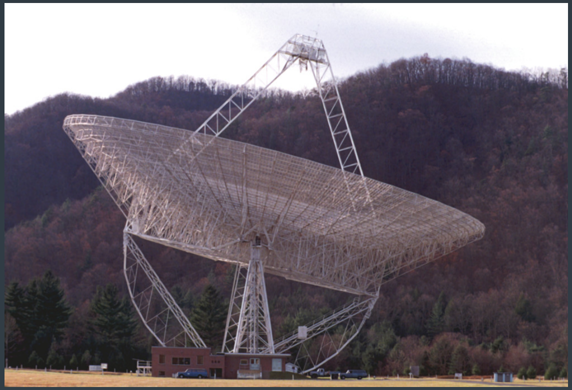
- Frederic Gueth IRAM school
- Uli Klein lecture

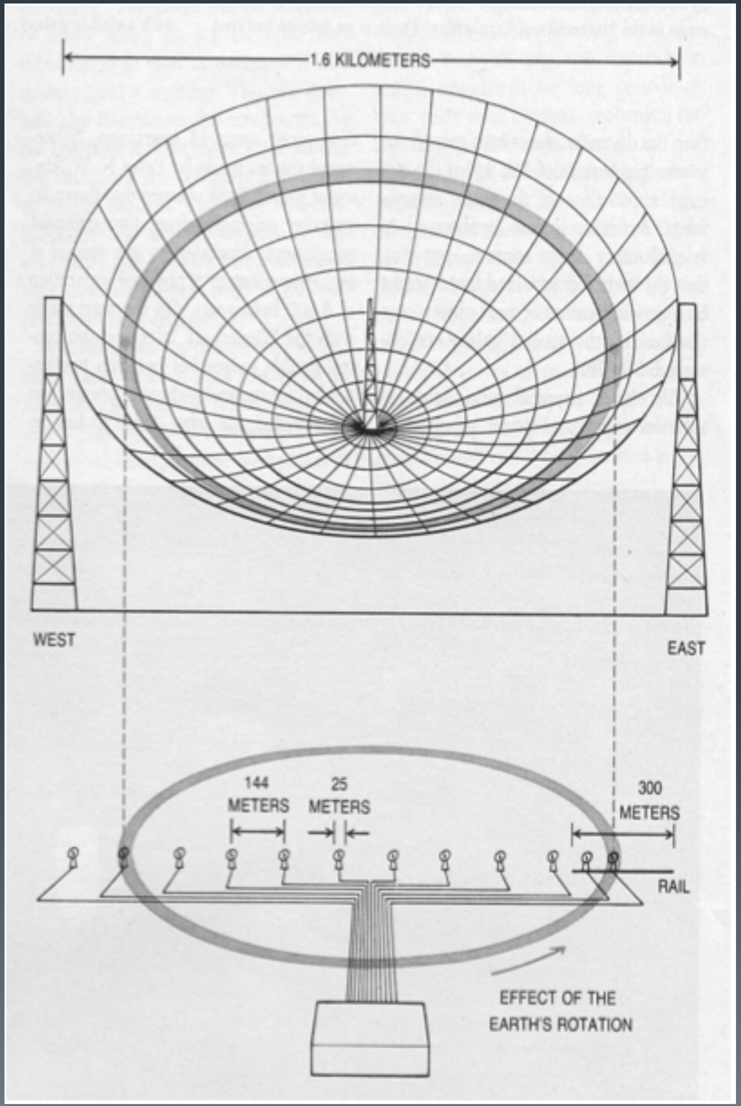
Lecture 4

- 2- element interferometer
- visibilities
- correlator
- uv-coverage
- synthesis imaging

Why Interferometry

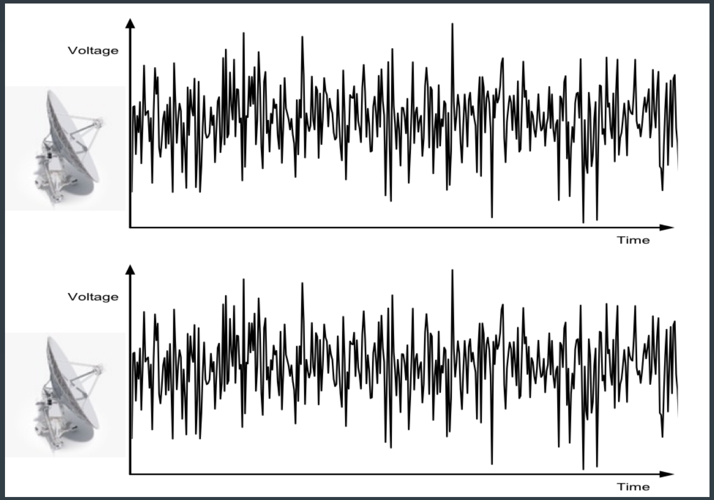


because there are limits



The old 300-foot transit telescope in Green Bank.

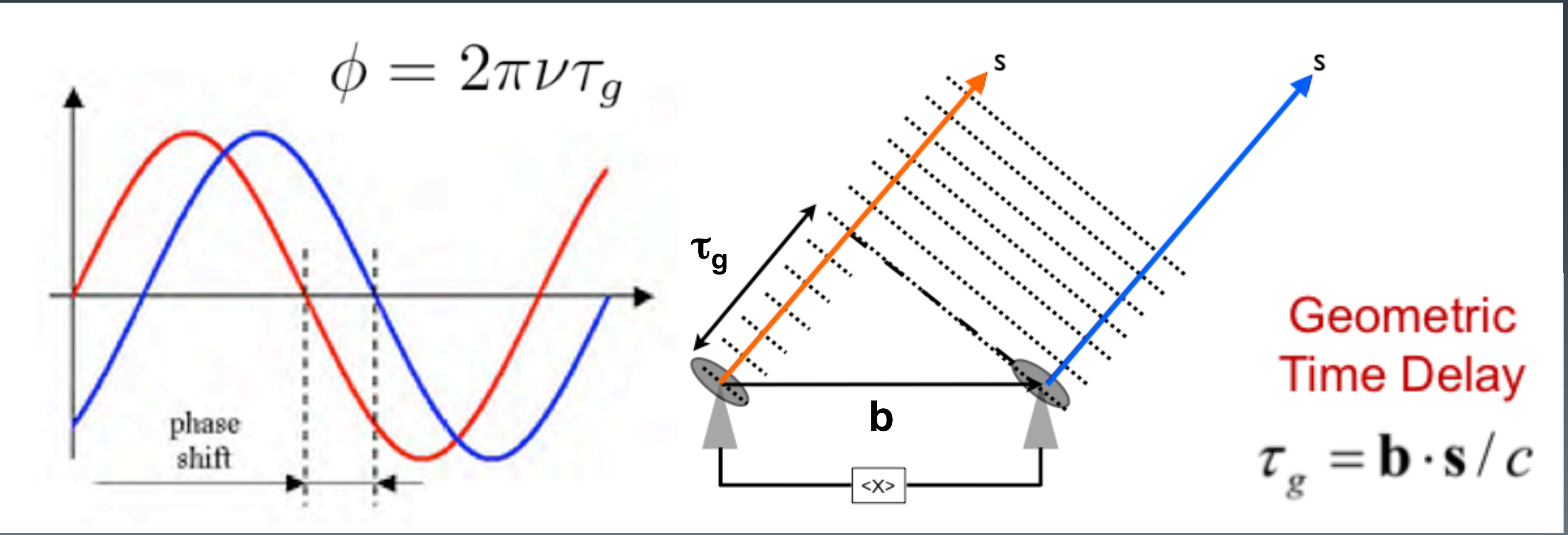
The basic two element interferometer



assume the following system

- monochromatic
- stationary reference frame
- RF throughout (it should be baseband)

Geometric delay τ_g

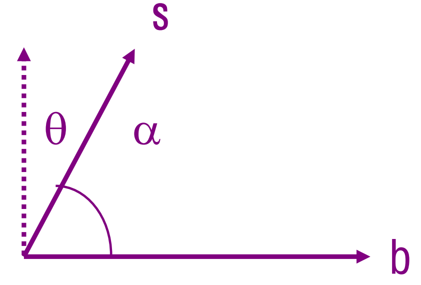


geometric delay some details

- To illustrate the response, expand the dot product in one dimension:

$$\frac{\mathbf{b} \bullet \mathbf{s}}{\lambda} = u \cos\alpha = u \sin\theta = ul$$

- Here, **u = b/λ is the baseline length in wavelengths**, and θ is the angle w.r.t. the plane perpendicular to the baseline.
- $l = \cos\alpha = \sin\theta$ is the direction cosine



- Consider the response R_c, as a function of angle, for two different baselines with u = 10, and u = 25 wavelengths:

$$R_c = \cos(20\pi l)$$

all sky fringe function

- Top: u = 10

$R_c = \cos(20\pi l)$

There are 20 whole fringes over the hemisphere.

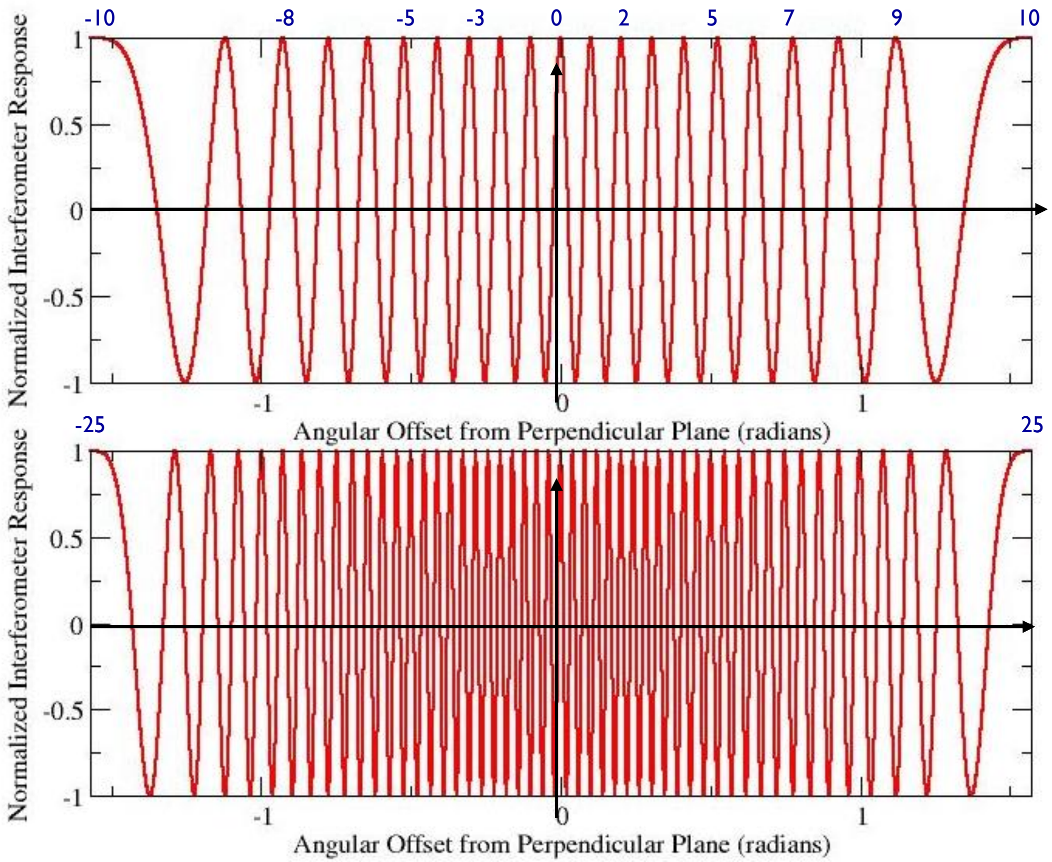
Peak separation 1/10 radians

- Bottom: u = 25

$R_c = \cos(50\pi l)$

There are 50 whole fringes over the hemisphere.

Peak separation 1/25 radians.



all sky fringe function [angular perspective]

Top Panel:

The absolute value of the response for u = 10, as a function of angle.

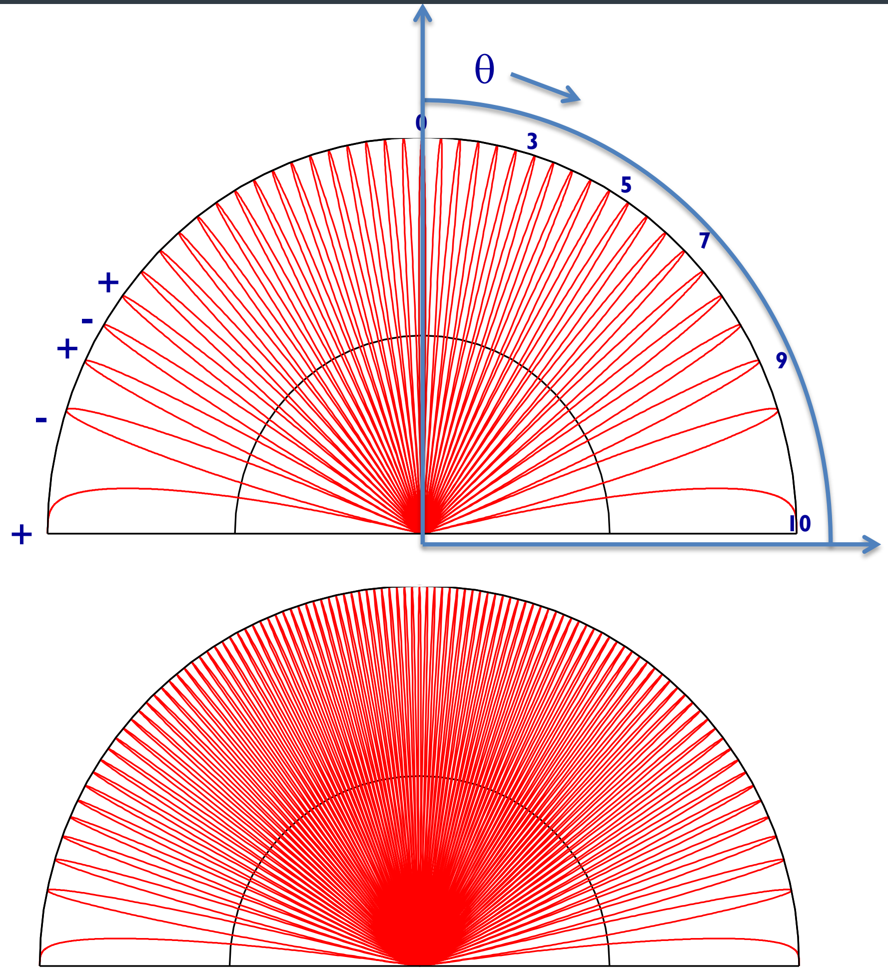
The ‘lobes’ of the response pattern alternate in sign.

Bottom Panel:

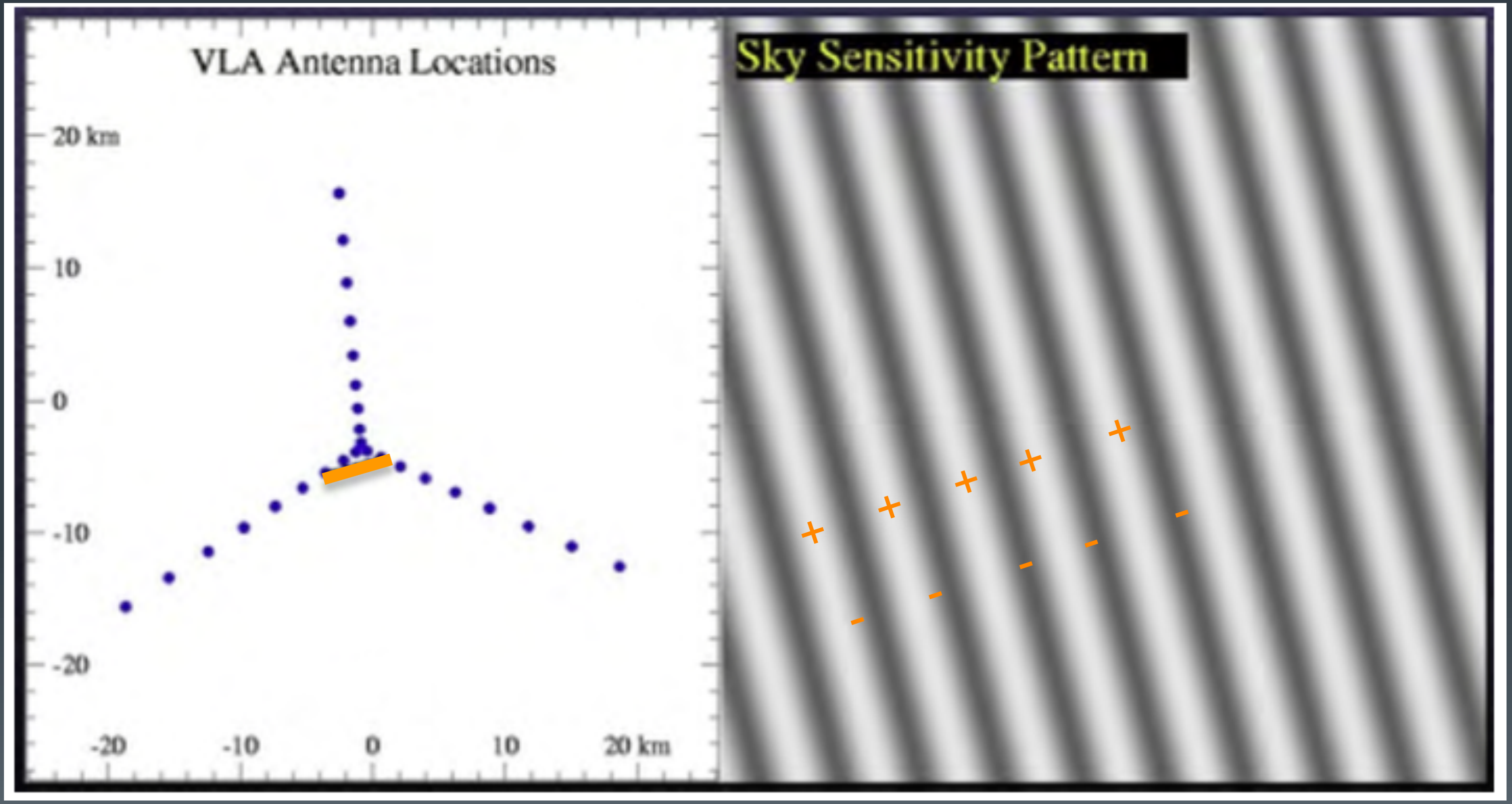
The same, but for u = 25.

Angular separation between lobes (of the same sign) is

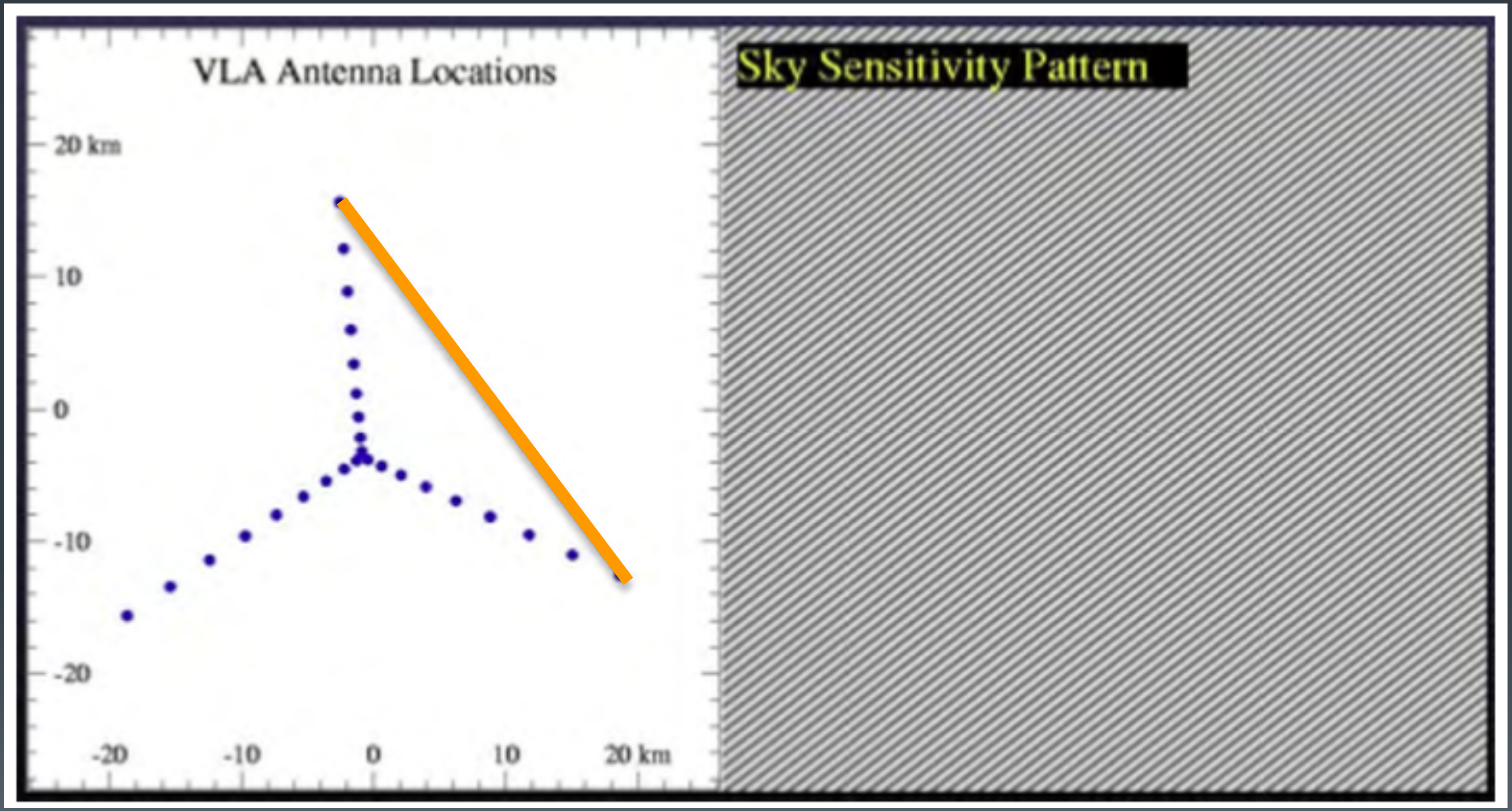
$\delta\theta \sim 1/u = \lambda/b$ radians.



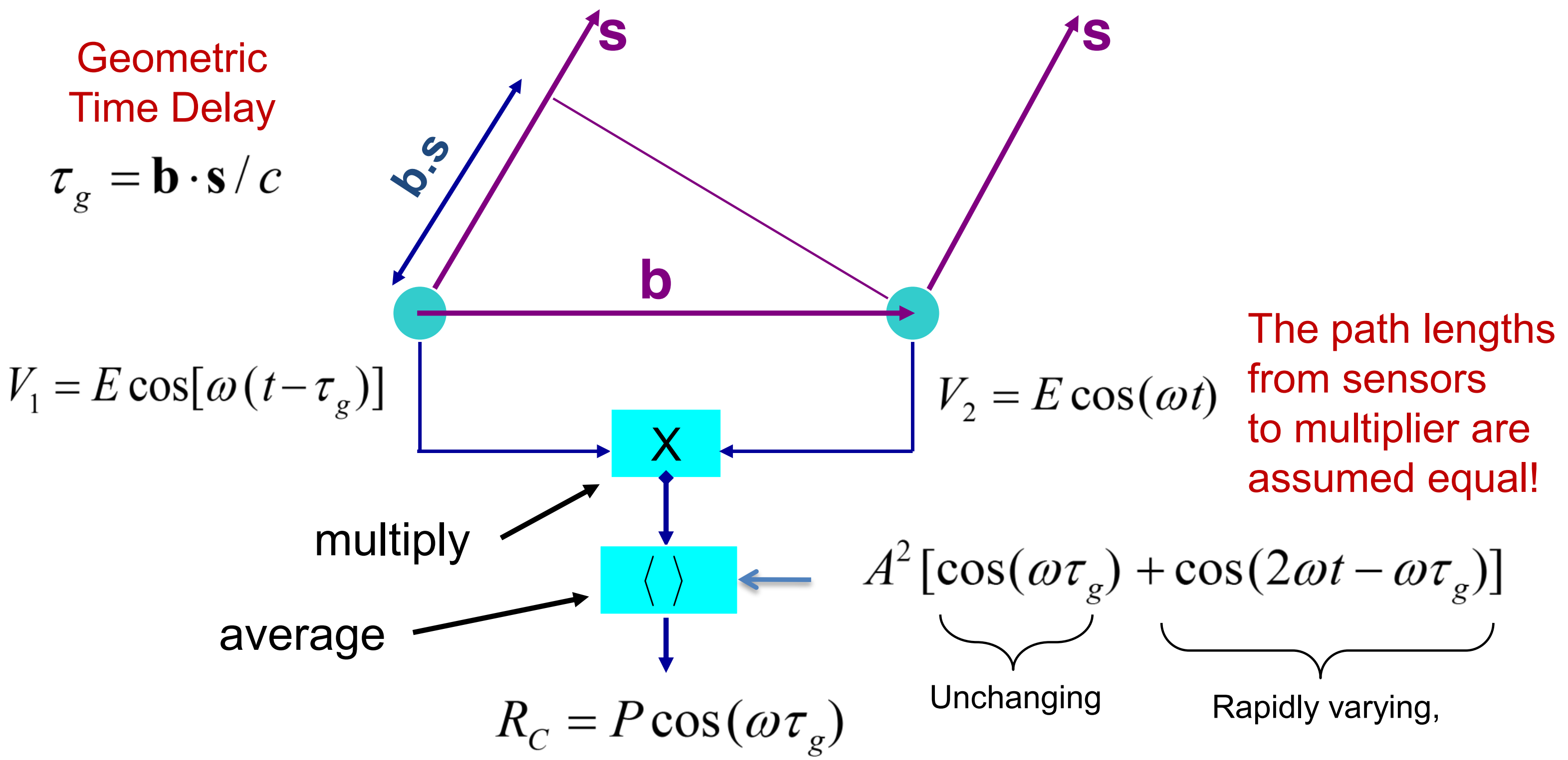
sky fringe function [different perspective]



sky fringe function [different perspective]



the measured fringe (COS correlator [even])



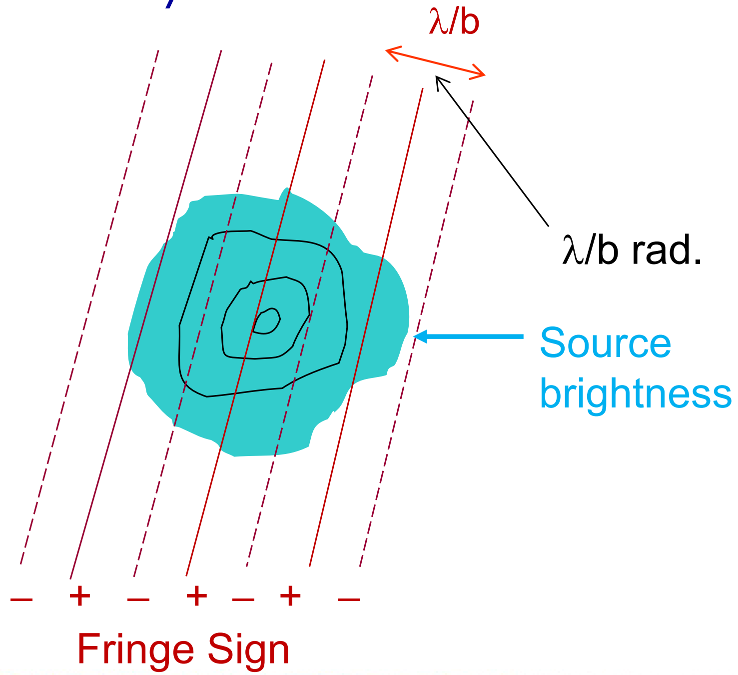
$$R_C = P\cos(\omega\tau_g) = P\cos\left(2\pi \frac{\mathbf{b} \cdot \mathbf{s}}{\lambda}\right)$$

the schematic overview

- The correlator can be thought of 'casting' a cosinusoidal coherence pattern, of angular scale $\sim\lambda/b$ radians, onto the sky.

- The correlator multiplies the source brightness by this coherence pattern, and integrates (sums) the result over the sky.

- Orientation set by baseline geometry.
- Fringe separation set by (projected) baseline length and wavelength.
  - Long baseline gives close-packed fringes
  - Short baseline gives widely-separated fringes
- Physical location of baseline unimportant, provided source is in the far field.



van Cittert-Zernike theorem

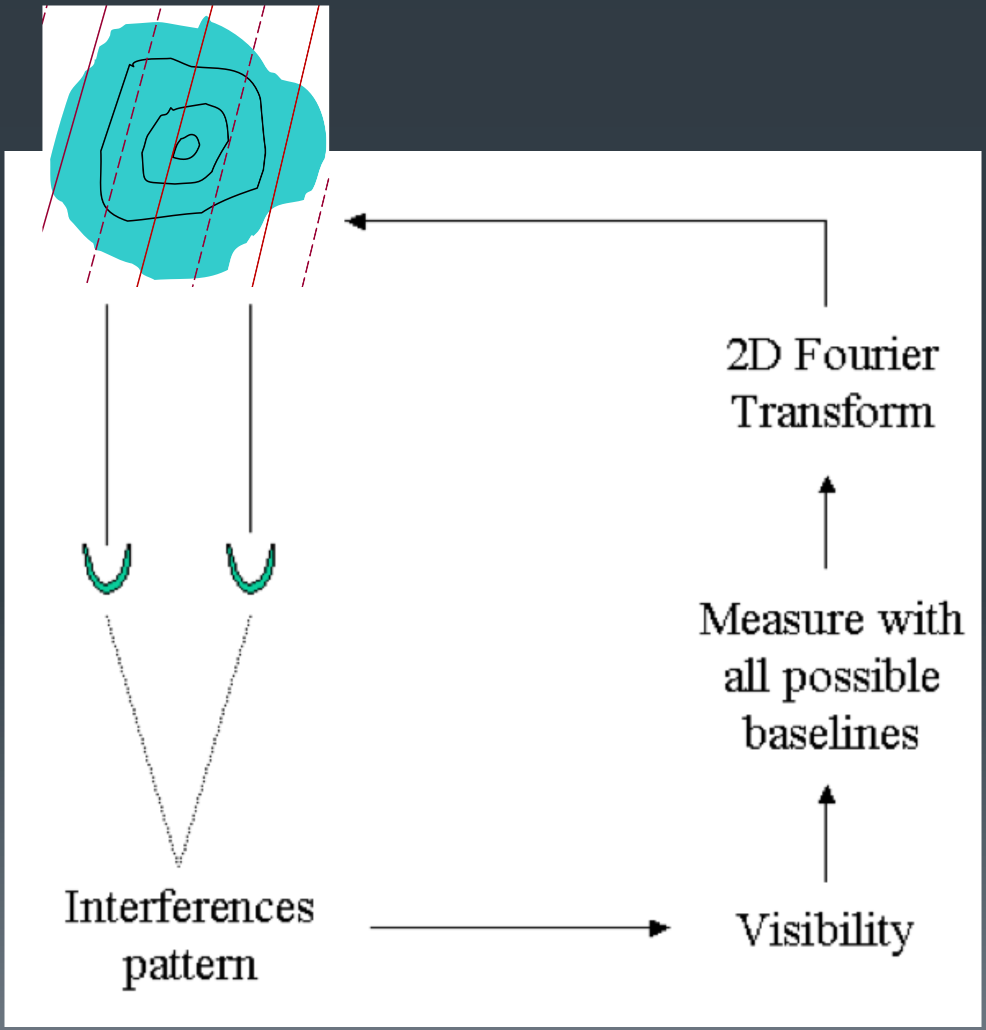
- **van Cittert–Zernike theorem (spatial)**
  - spatial autocorrelation of $S(x) = \mathrm{FT}(\text{brightness})$

    $S(x_1)\, S(x_2) = \Sigma(u) \rightleftharpoons S(\alpha)$
  - implementation: aperture synthesis

- Wiener-Kichnine theorem **(temporal)**
  - temporal autocorrelation of $S(t) = \mathrm{FT}(\text{spectra})$

    $S(t_1)\, S(t_2) = \Sigma(\tau) \rightleftharpoons S(\nu)$
  - implementation: FT spectrometers

van Cittert-Zernike theorem



Integration over the entire sky and averaging in time

The response from an extended source is obtained by summing the responses at each antenna to all the emission over the sky, multiplying the two, and averaging:

$$R_C = \left\langle \iint V_1 d\Omega_1 \times \iint V_2 d\Omega_2 \right\rangle$$

The averaging and integrals can be interchanged and, **providing the emission is spatially incoherent**, we get

$$R_C = \iint I_\nu(\mathbf{s}) \cos(2\pi\nu \mathbf{b}\cdot\mathbf{s}/c)\, d\Omega$$

This expression links what we want – the source brightness on the sky, $I_\nu(\mathbf{s})$, – to something we can measure - R_C, the interferometer response.

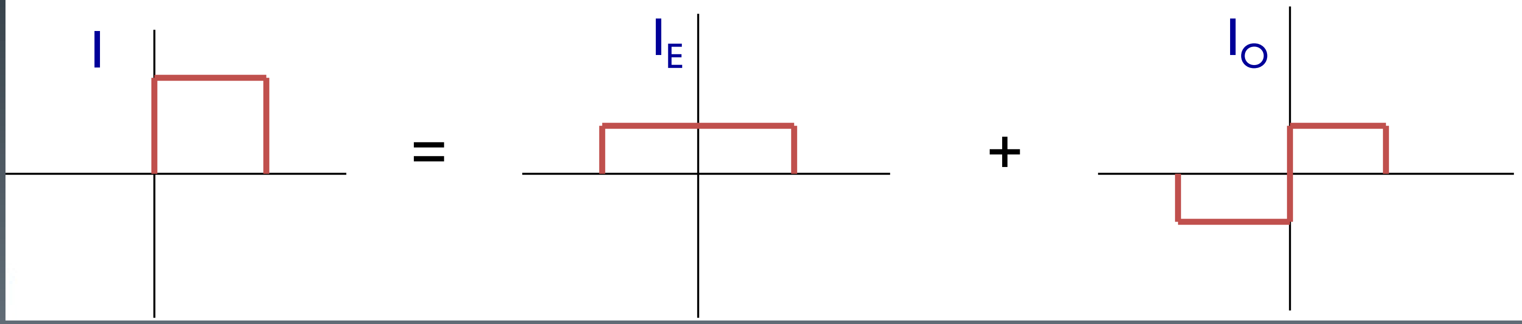
detour even and odd functions

Any real function, I(x,y), can be expressed as the sum of two real functions which have specific symmetries:

$$I(x,y) = I_E(x,y) + I_O(x,y)$$

An even part: $$I_E(x,y) = \frac{I(x,y) + I(-x,-y)}{2} = I_E(-x,-y)$$

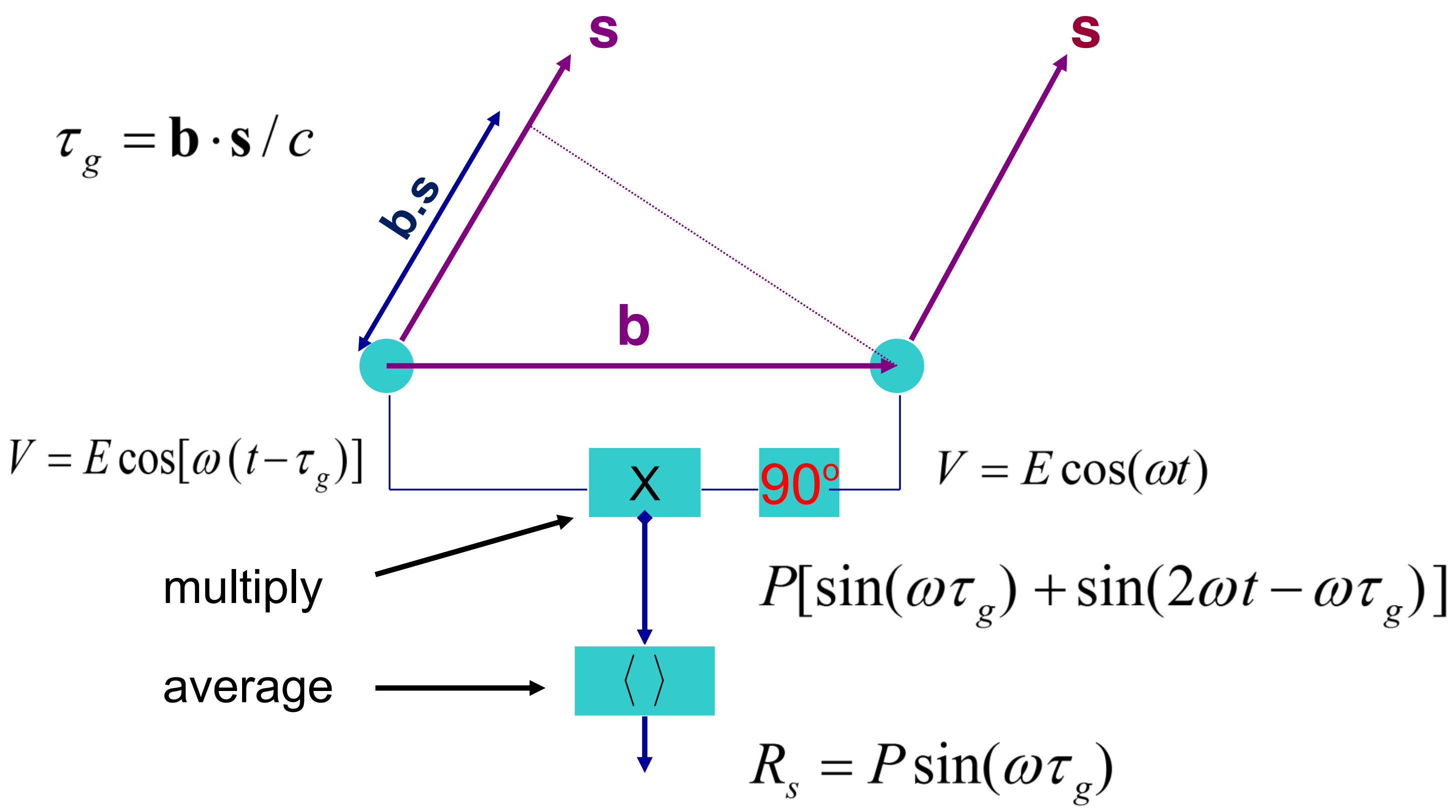
An odd part: $$I_O(x,y) = \frac{I(x,y) - I(-x,-y)}{2} = -I_O(-x,-y)$$



The cosine fringe pattern is even, the response R_c of our interferometer to an odd brightness $-I_o(-x,-y)$ distribution is 0 !!!

making a SIN correlator [odd]

We generate the 'sine' pattern by inserting a 90 degree phase shift in one of the signal paths.



Complex Visibilities

The correlator is called complex if it produces Cosine and Sine fringes.

- We now DEFINE a complex function, the complex visibility, V, from the two independent (real) correlator outputs R_C and R_S:

$$V = R_C - iR_S = Ae^{-i\phi}$$

phase errors are odd features in the images

where

$$A = \sqrt{R_C^2 + R_S^2}$$

amplitude errors are even features in the images

$$\phi = \tan^{-1}\left(\frac{R_S}{R_C}\right)$$

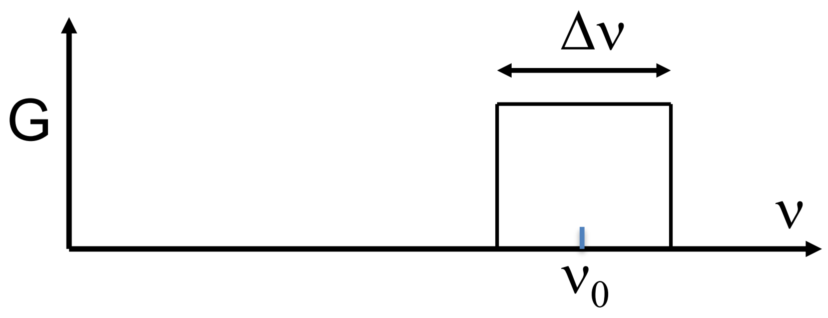
- This gives us a beautiful and useful relationship between the source brightness, and the response of an interferometer:

$$V_\upsilon(\mathbf{b}) = R_C - iR_S = \iint I_\nu(s) e^{-2\pi i \nu \mathbf{b}\cdot\mathbf{s}/c} d\Omega$$

- With the right geometry, this is a 2-D Fourier transform, giving us a well established way to recover $I(\mathbf{s})$ from $V(\mathbf{b})$.

move from monochromatic assumption

- Real interferometers must accept a range of frequencies. So we now consider the response of our interferometer over frequency.
- Define the frequency response functions, $G(\nu)$, as the amplitude and phase variation of the signal over frequency.



- The function $G(\nu)$ is primarily due to the gain and phase characteristics of the electronics, but can also contain propagation path effects.
- In principle, $G(\nu)$ is a complex function.

the effect of bandwidth

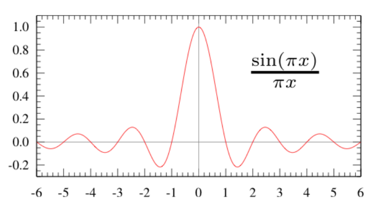
- To find the finite-bandwidth response, we integrate our fundamental response over a frequency width $\Delta\nu$, centered at ν_0:

$$V = \int \left(\frac{1}{\Delta\nu} \int_{\upsilon_0 - \Delta\upsilon/2}^{\upsilon_0 + \Delta\upsilon/2} I(\mathbf{s}, \upsilon) G_1(\upsilon) G_2^*(\upsilon) e^{-i2\pi\upsilon\tau_g} d\upsilon \right) d\Omega$$

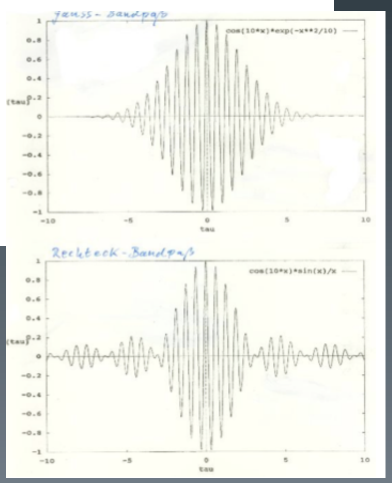
- If the source intensity does not vary over the bandwidth, and the instrumental gain parameters G_1 and G_2 are square and identical, then

$$V = \iint I_\upsilon(\mathbf{s}) \frac{\sin(\pi\tau_g\Delta\upsilon)}{\pi\tau_g\Delta\upsilon} e^{-2i\pi\nu_0\tau_g} d\Omega = \iint I_\upsilon(\mathbf{s}) \mathrm{sinc}(\tau_g\Delta\upsilon) e^{-2i\pi\nu_0\tau_g} d\Omega$$

where the **fringe attenuation function**, sinc(x), is defined as:

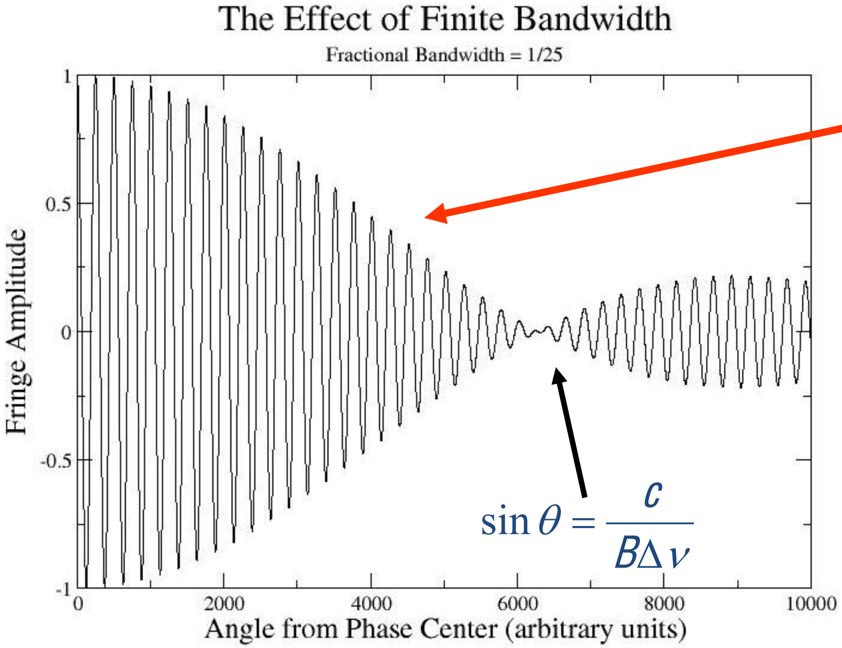


$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



the effect of bandwidth

- For a square bandpass, the bandwidth attenuation reaches a null when $\tau_g \Delta\nu = 1$, or $\sin\theta = \lambda / B \big/ \Delta\upsilon / \upsilon_0 = \dfrac{c}{B\Delta\nu}$
- For the old VLA, and its 50 MHz bandwidth, and for the 'A' configuration, the null was ~1.3 degrees away.
- For the EVLA, $\Delta\nu$ = 2 MHz, and B = 35 km, then the null occurs at about 27 degrees off the meridian.



Fringe Attenuation function:

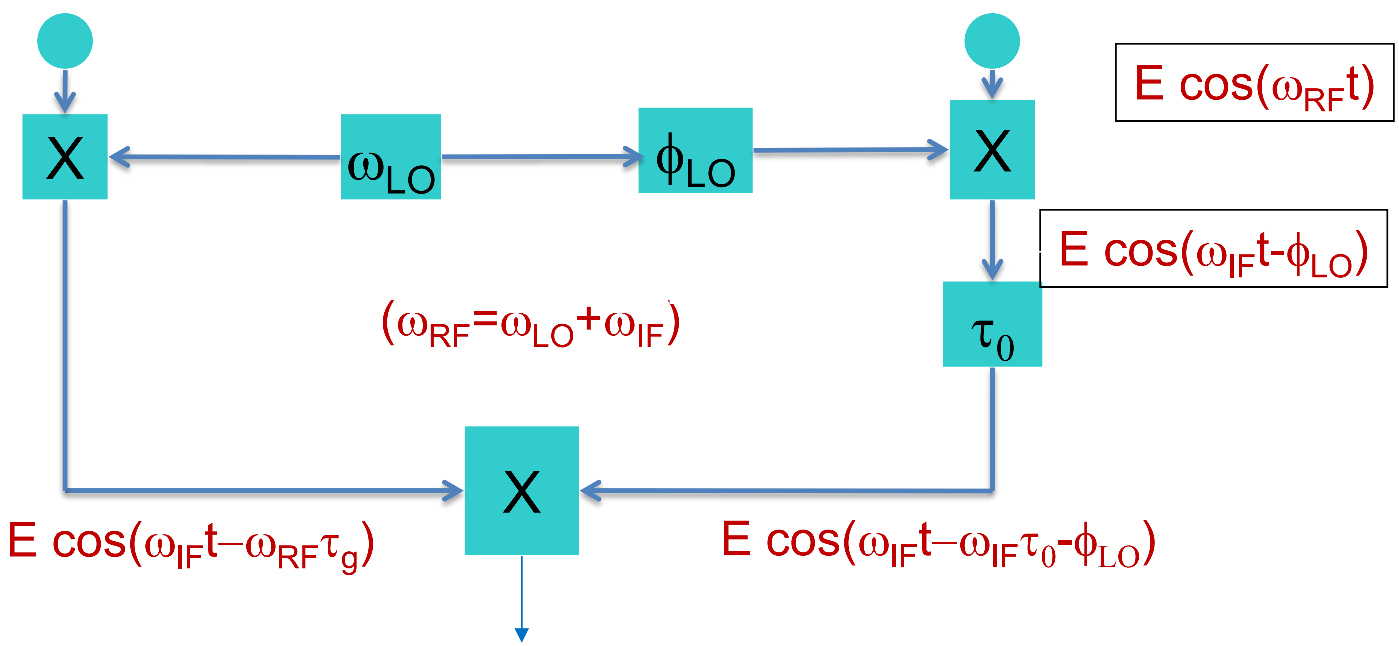
$$\mathrm{sinc}\left(\frac{B}{\lambda}\frac{\Delta\upsilon}{\upsilon}\sin\theta\right) = \mathrm{sinc}\left(\frac{B\Delta\upsilon}{c}\sin\theta\right)$$

Note: The fringe-attenuation function depends only on bandwidth and baseline length – not on frequency.

move to the IF signal

relation LO down conversion (see also last lecture)

- The RF signals are multiplied by a pure sinusoid, at frequency ν_{LO}
- We can add arbitrary phase ϕ_{LO} on one side.



$$V = E^2 e^{-i(\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO})}$$

IF signal: recover the correct visibility phase

- The correct phase (RF interferometer) is: $\omega_{RF}\left(\tau_g - \tau_0\right)$
- The observed phase (with frequency downconversion) is:

$$\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO}$$

- These will be the same when the LO phase is set to:

$$\phi_{LO} = \omega_{LO}\tau_0$$

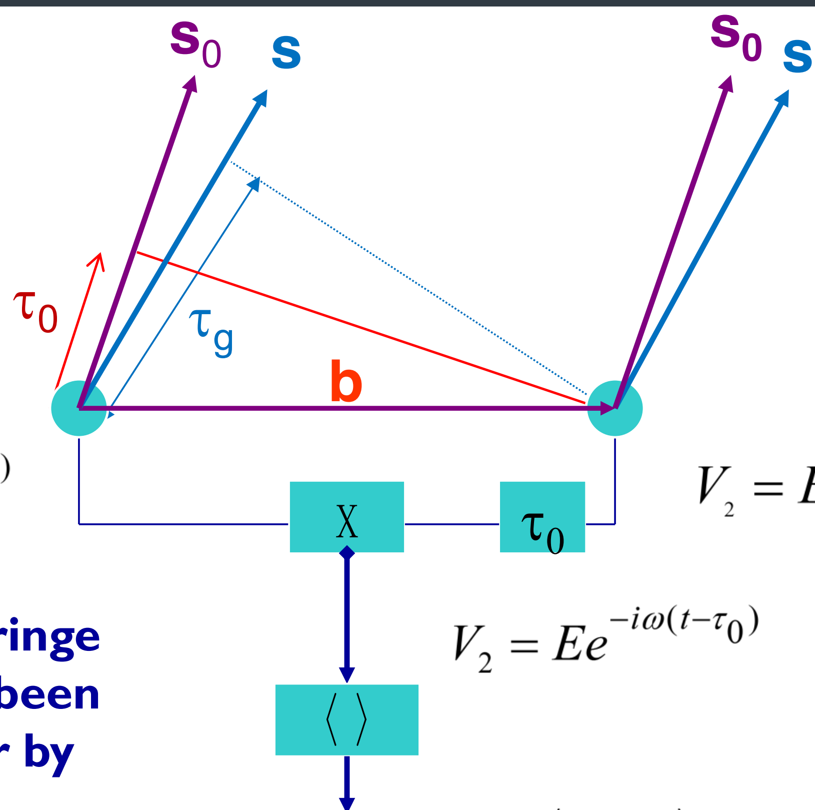
- This is necessary because the delay, τ_0, has been added in the IF portion of the signal path, rather than at the frequency at which the delay actually occurs.
- The phase adjustment of the LO compensates for the delay having been inserted at the IF , rather than at the RF.

an interferometer is not stationary

adding a time delay needs to be done continuously

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$

$$\tau_0 = \mathbf{b} \cdot \mathbf{s}_0 / c$$



S0 = reference (delay) direction

S = general direction

$$V_1 = Ee^{-i\omega(t-\tau_g)}$$

$$V_2 = Ee^{-i\omega t}$$

$$V_2 = Ee^{-i\omega(t-\tau_0)}$$

The entire fringe pattern has been shifted over by angle

sin θ = cτ_0/b

$$V = \langle V_1 V_2^* \rangle = E^2 e^{-i[\omega(\tau_0 - \tau_g)]}$$

$$= E^2 e^{i2\pi[\upsilon \mathbf{b} \cdot (\mathbf{s} - \mathbf{s}_0)/c]}$$

the three centres of interferometer

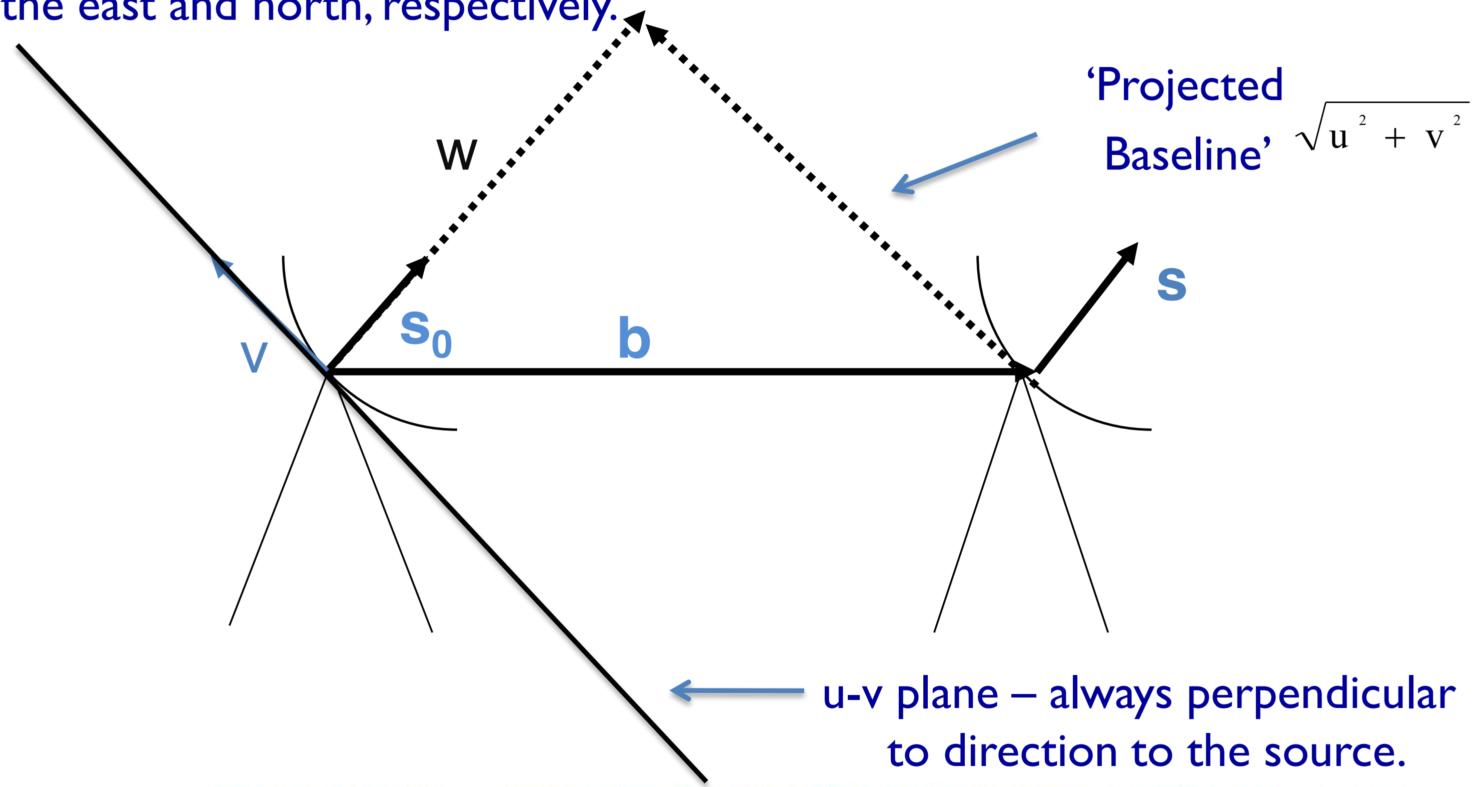
1. **Beam Tracking (Pointing) Center**: Where the antennas are pointing to. (Or, for phased arrays, the phased array center position).
2. **Delay Tracking Center:** The location for which the delays are being set for maximum wide-band coherence.
3. **Phase Tracking Center:** The location for which the LO phase is slipping in order to track the coherence pattern.

- Note: Generally, we make all three the same. #2 and #3 are the same for an 'RF' interferometer. They are separable in a LO downconversion system.

the general coordinate system

Interferometer ONLY does not cover the position of the individual telescopes

- This is the coordinate system in most general use for synthesis imaging.
- **w** points to, and follows the source, **u** towards the east, and **v** towards the north celestial pole. The direction cosines *l* and m then increase to the east and north, respectively.



the direction cosines

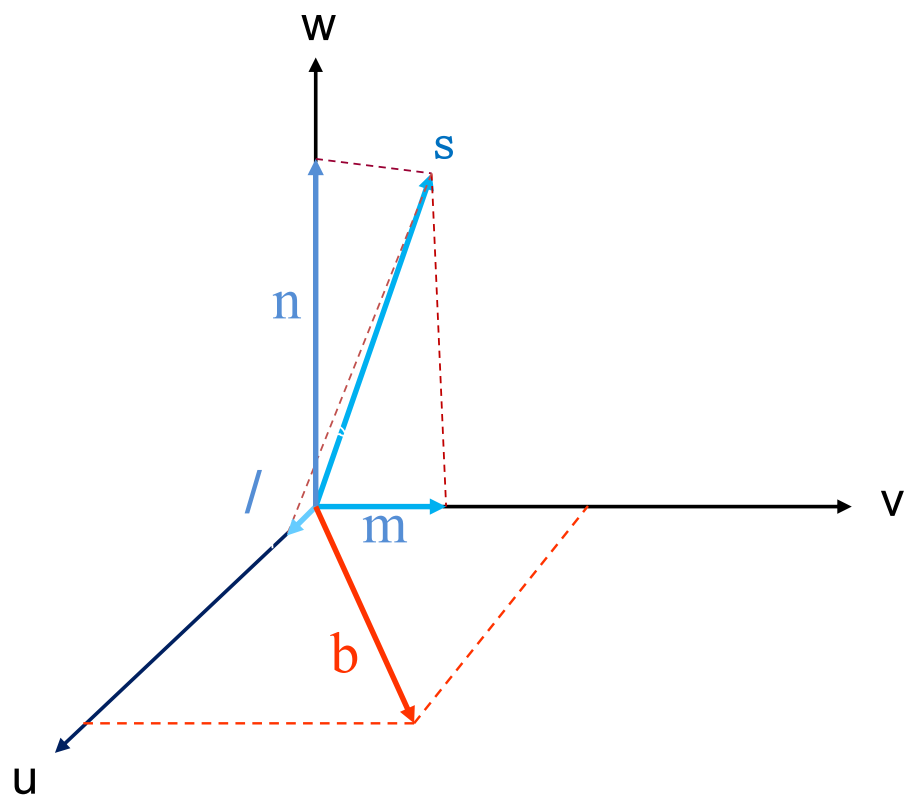
source only describing the vector S in the (l,m,n) system

The unit direction vector **s** is defined by its projections (l,m,n) on the (u,v,w) axes. These components are called the **Direction Cosines**.

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$



The angles, α, β, and θ are between the direction vector and the three axes.

3d visibility

- What if the interferometer does not measure the coherence function on a plane, but rather does it through a volume? In this case, we adopt a different coordinate system. First we write out the full expression:

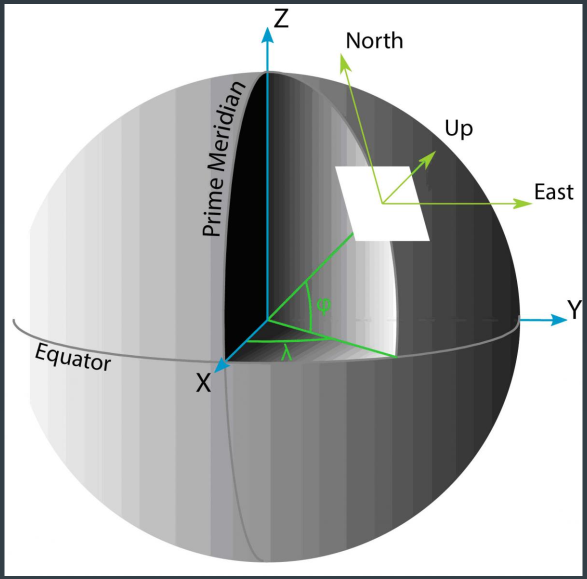
$$V_\nu(\mathrm{u},\mathrm{v},\mathrm{w}) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi(\mathrm{u}l+\mathrm{v}m+\mathrm{w}n)} dldm$$

(Note that this is not a 3-D Fourier Transform).

- We orient the w-axis of the coordinate system to point to the region of interest. The u-axis point east, and the v-axis to the north celestial pole.
- We introduce phase tracking, so the fringes are 'stopped' for the direction l=m=0. This means we adjust the phases by $e^{2i\pi w}$
- Then, remembering that $n^2 = 1 - l^2 - m^2$ we get:

$$V_\nu(u,v,w) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} dldm$$

antenna position coordinate system



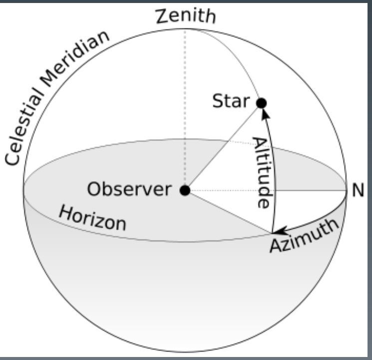
XYZ – coordinate system for position on the earth
- based on a model for the earth
- essentially your correlator knows all about it
- careful if the position of the telescope is wrong

set the system fixed to the terrestrial system:

X hour angle 0^h and declination 0^o

Y hour angle -6^h and declination 0^o

Z declination 90^o



Horizontal coordinate system

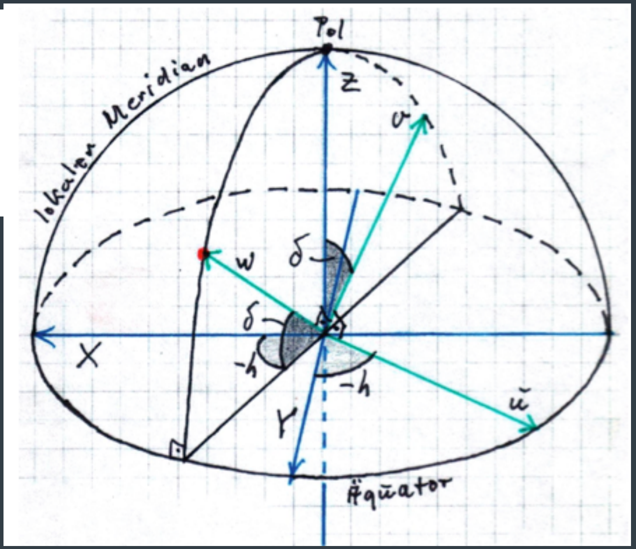
The baseline separation (L_x, L_y, L_z) to a reference antenna.

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = D \cdot \begin{pmatrix} \cos\phi \cdot \sin E - \sin\phi \cdot \cos E \cdot \cos A \\ \cos E \cdot \sin A \\ \sin\phi \cdot \sin E + \cos\phi \cdot \cos E \cdot \cos A \end{pmatrix}$$

geographic latitude ϕ, elevation E, azimuth A

baselines in uvw coordinates

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \sin h & \cos h & 0 \\ -\sin\delta \cdot \cos h & \sin\delta \cdot \sin h & \cos\delta \\ \cos\delta \cdot \cos h & -\cos\delta \cdot \sin h & \sin\delta \end{pmatrix} \cdot \begin{pmatrix} L_X \\ L_Y \\ L_Z \end{pmatrix}$$

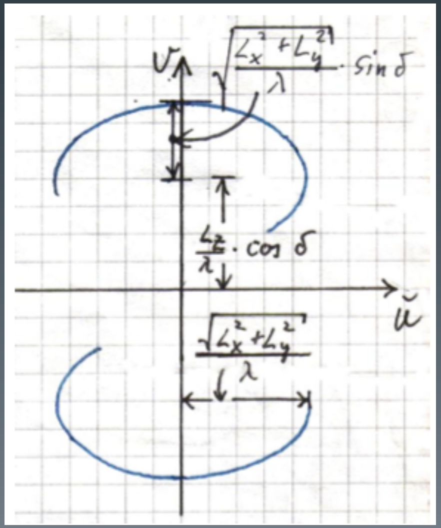


$$\begin{aligned} u &= \frac{1}{\lambda} \cdot (\sin h \cdot L_X + \cos h \cdot L_Y) \\ v &= \frac{1}{\lambda} \cdot (-\sin\delta \cdot \cos h \cdot L_X + \sin\delta \cdot \sin h \cdot L_Y + \cos\delta \cdot L_Z) \end{aligned}$$

eliminate h

$$u^2 + \left[\frac{v - \left(\frac{L_Z}{\lambda}\right) \cdot \cos\delta}{\sin\delta} \right]^2 = \frac{L_X^2 + L_Y^2}{\lambda^2}$$

UV coverage



visibility shopping list

- measure a visibility at a specific time towards a source at declination δ and right ascension α with respect to the phase centre
- get the UV coordinates for each measurement (need the antenna position for this)

$$V_\nu(u,v,w) = \iint \frac{I_\nu(l,m)}{\sqrt{1-l^2-m^2}} e^{-2i\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} dldm$$

observational parameters declination right ascension

$$\delta = arcsin(m \cdot cos\delta_0 + sin\delta_0 \cdot \sqrt{1-l^2-m^2})$$

$$\alpha = \alpha_0 + arctan(\frac{l}{cos\delta_0 \cdot \sqrt{1-l^2-m^2} - m \cdot sin\delta_0})$$

$$l = cos\delta \cdot sin(\alpha - \alpha_0)$$

$$m = sin\delta \cdot cos\delta_0 - cos\delta_0 \cdot sin\delta_0 \cdot cos(\alpha - \alpha_0)$$

get the real Fourier transformation transform 3d to 2d

- The expression is still not a proper Fourier transform.
- We can get a 2-d FT if the third term in the phase factor is sufficient small.

- The third term in the phase can be neglected if it is much less than unity:

$$w\left[1-\sqrt{1-l^2-m^2}\right]=w(1-\cos\theta)\sim w\theta^2/2<<1$$

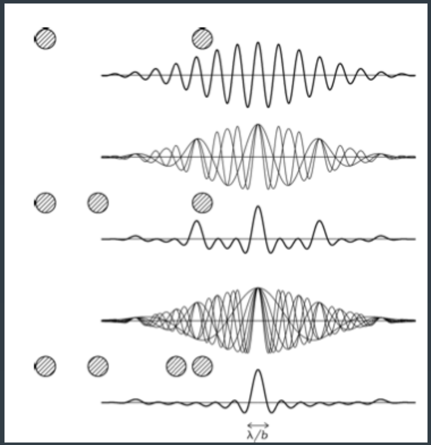
- This condition holds when:
  (angles in radians!)

$$\theta_{\max}<\sqrt{\frac{1}{2\mathrm{w}}}\sim\sqrt{\frac{\lambda}{\mathrm{B}}}\sim\sqrt{\theta_{syn}}$$

- If this condition is met, then the relation between the Intensity and the Visibility again becomes a 2-dimensional Fourier transform:

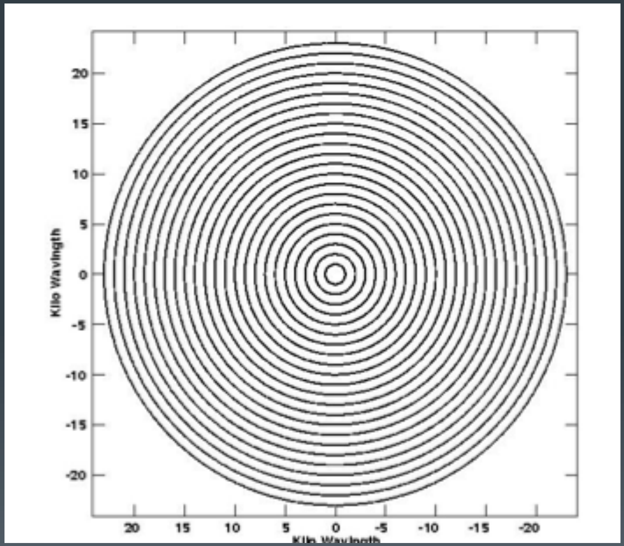
$$V_\nu^{'}(u,v)=\iint I_\nu(l,m)e^{-2i\pi(ul+vm)}dldm$$

lets get more antennas

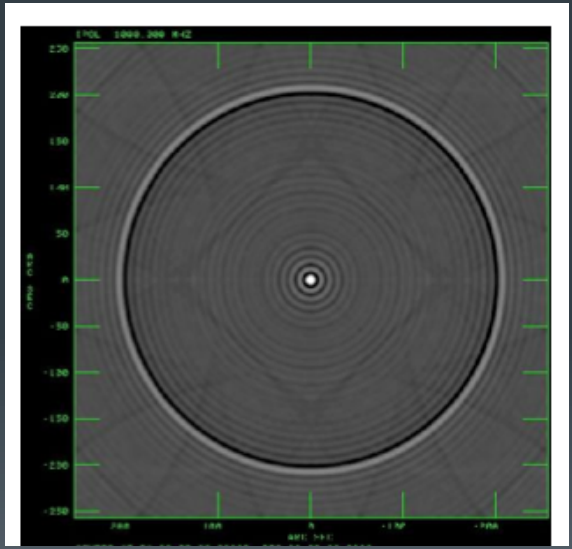


Fourier Transformation

12 hours UV coverage $\delta = 90^o$

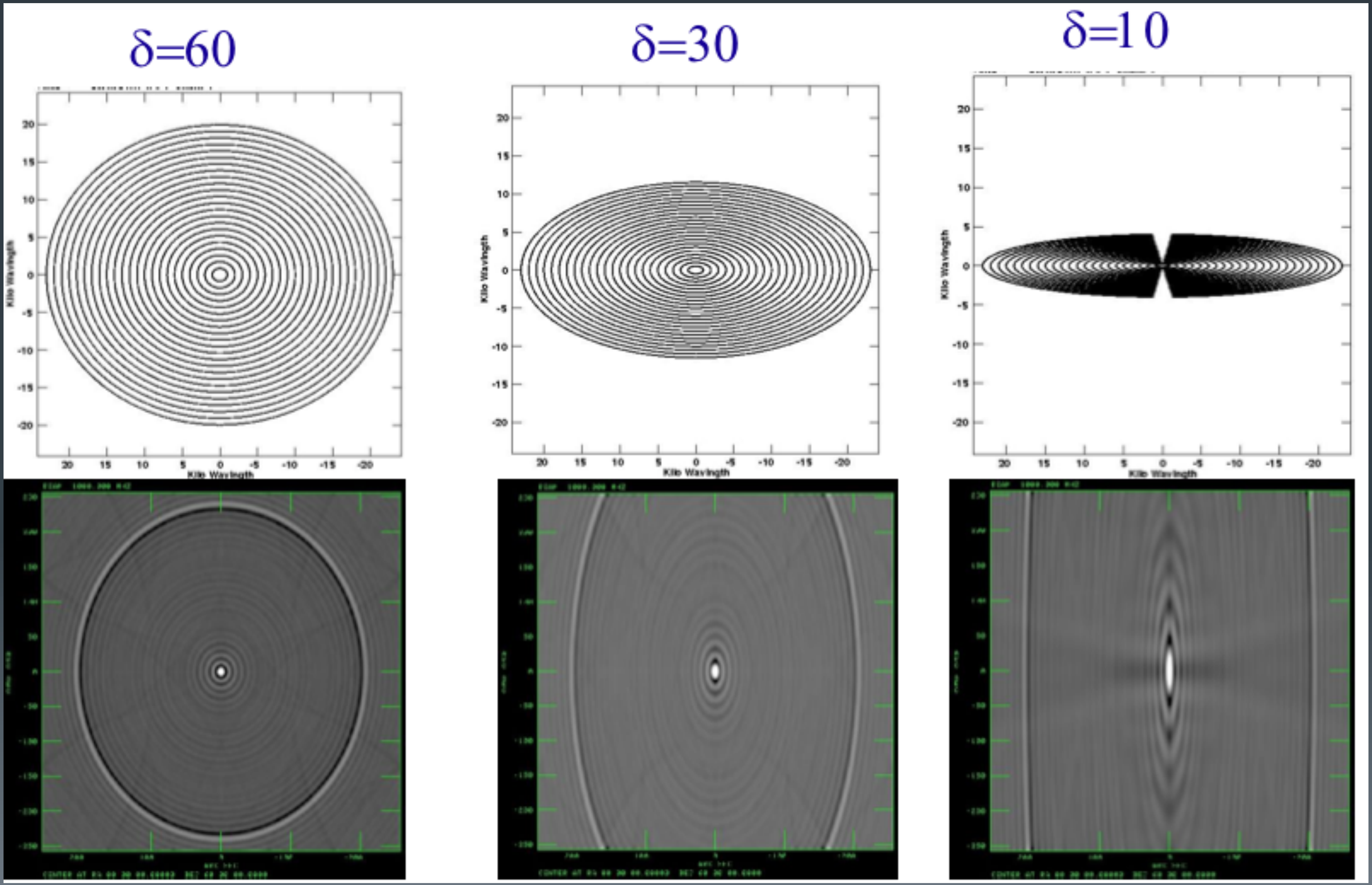


synthesized beam



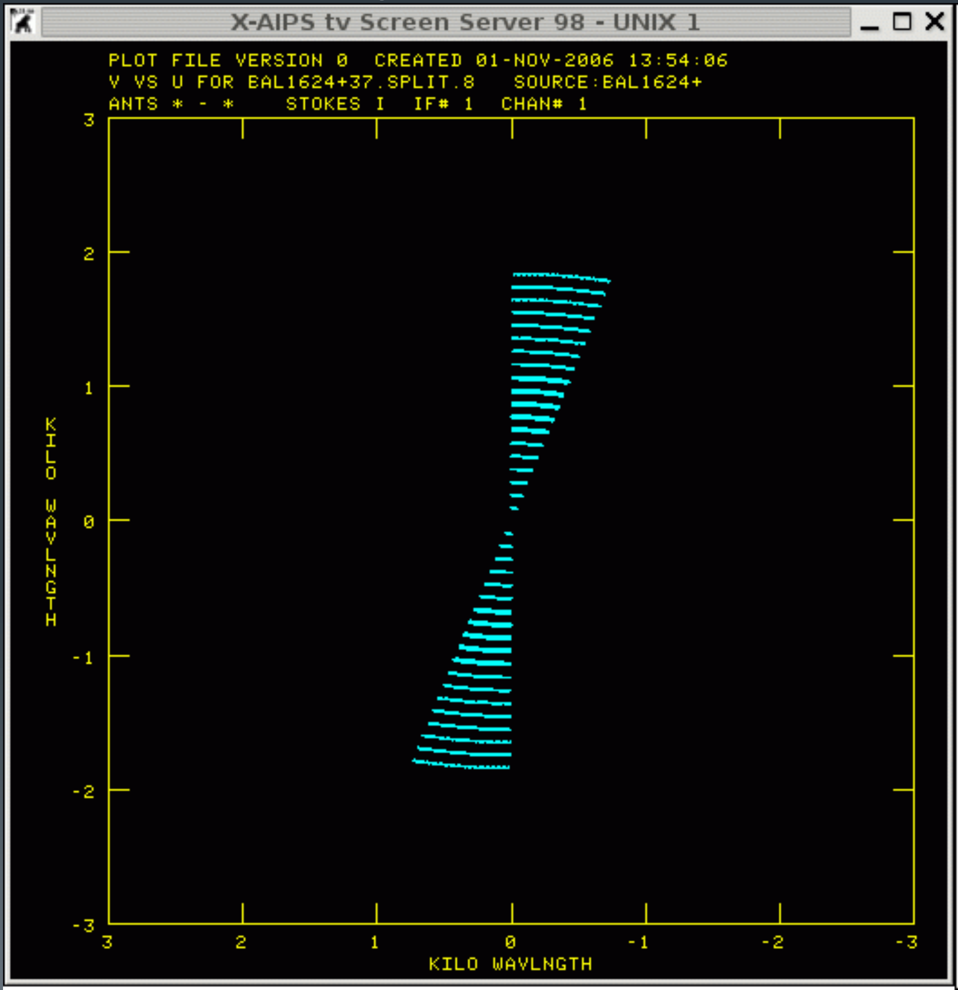
only use uv points and not the measured values of the visibility

UV coverage and synthesised beams



Synthesis Imaging

UV coverage



brightness distribution