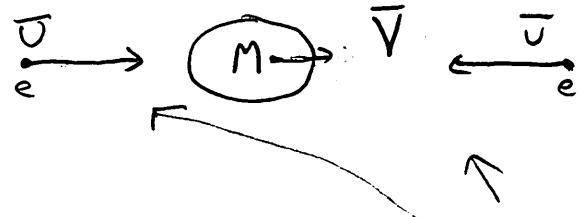
Following Melia

Particle acceleration.

Second order Fermi, scattering off magnetic irregularities. 1-D approach



Δ = average separation between M and e.

collision time for a head-on collision $\tau_h = \dfrac{\Delta}{U+V}$

-//- catch-up collision $\tau_c = \dfrac{\Delta}{U-V}$

Probability for a head-on collision

$$P_h = \frac{1/\tau_h}{1/\tau_h + 1/\tau_c} = \frac{U+V}{2U} \quad . \text{ Then } P_c = \frac{U-V}{2U}$$

Consider elastic collisions : $U_i' = U_f'$ in the CM frame

head on $\quad U_i' = -(U+V) \Rightarrow U_a' = +(U+V) \Rightarrow U_a = +(U+V) + V = +(U+2V)$

catch up $\quad U_i' = U - V \Rightarrow U_a' = -(U-V) \Rightarrow U_a = -(U-V) + V = -(U-2V)$

Energy change

$$\Delta E_h = \frac{1}{2} m (U+2V)^2 - \frac{1}{2} m U^2 = \frac{1}{2} m \left(U^2 + 4V^2 + 4UV - U^2\right) = \underline{2mV(U+V)}$$

$$\Delta E_c = \frac{1}{2} m (U-2V)^2 - \frac{1}{2} m U^2 = \frac{1}{2} m \left(U^2 + 4V^2 - 4UV - U^2\right) = \underline{-2mV(U-V)}$$

$$\langle \Delta E \rangle = P_h \Delta E_h + P_c \Delta E_c = \frac{U+V}{2U} 2mV(U+V) - \frac{U-V}{\cancel{2}U} 2mV(U-V)$$

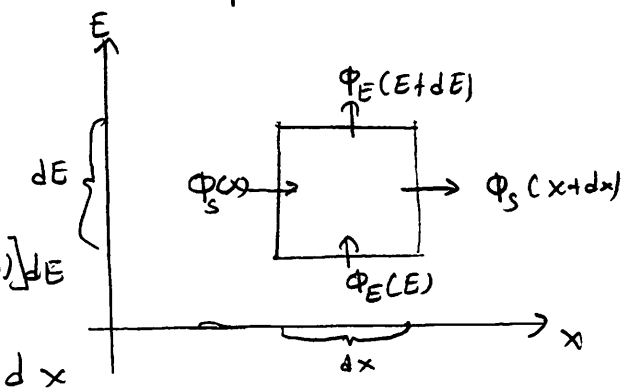
$$= \frac{mV}{U}\left[U^2 + V^2 + 2UV - U^2 - V^2 + 2UV\right] = \underline{4mV^2} \Rightarrow \frac{\langle \Delta E \rangle}{E} = \underline{8\left(\frac{V}{U}\right)^2}$$

If ν is the collision rate then $\frac{dE}{dt} = \langle \Delta E \rangle \nu = 8\nu \left(\frac{V}{U}\right)^2 E = \underline{aE}$

This is second order in $\frac{V}{U}$ does not do much

How to get a power-law out of this.

First, motivate the diffusion-loss equation



$N(E, x, t)$: particle energy dist.

$$\frac{dN}{dt} dE\,dx = \left[\Phi_s(E,x,t) - \Phi_s(E,x+dx,t)\right] dE$$

$$+ \left[\Phi_E(E,x,t) - \Phi_E(E+dE,x,t)\right] dx$$

$$+ \frac{\partial N}{\partial t} dE\,dx$$

. Divide by $dx\,dE$ to get

$$\frac{dN}{dt} = -\frac{\partial \Phi_s}{\partial x} - \frac{\partial \Phi_E}{\partial E} + \frac{\partial N}{\partial t}$$

Spatial flux evolves diffusively: $\Phi_s = -D \frac{\partial N}{\partial x}$

So $\boxed{\frac{dN}{dt} = D \frac{\partial^2 N}{\partial x^2} - \frac{\partial \Phi_E}{\partial E} + \frac{\partial N}{\partial t}}$ in 3D $\frac{\partial^2 N}{\partial x^2} \longrightarrow \nabla^2 N$

Simplify: mimic diffusion with escape after $\langle t_{inc} \rangle = \tau$

Set $D = 0$ $\quad \frac{\partial N}{\partial t} = \frac{N}{\tau}$

$$\frac{dN}{dt} = -\frac{\partial \Phi_E}{\partial E} + \frac{N}{\tau}$$

Now $\Phi_E = N \frac{dE}{dt} = N a E$

So $\frac{dN}{dt} = -\frac{\partial}{\partial E}(NaE) - \frac{N}{\tau}$

In steady-state $\frac{dN}{dt} = 0$ and $N = N(E)$ only

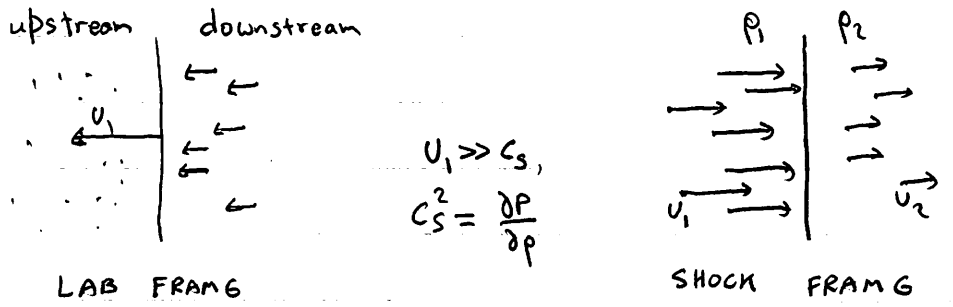
$$aN + aE\frac{\partial N}{\partial E} = -\frac{N}{\tau} \Rightarrow \frac{\partial N}{\partial E} = -\left(1 + \frac{1}{a\tau}\right)\frac{N}{E} \Rightarrow$$

$$\frac{\partial N}{N} = -\left(1 + \frac{1}{a\tau}\right)\frac{\partial E}{E} \Rightarrow \boxed{N = N_0 E^{-(1+\frac{1}{a\tau})} \quad \text{power law}}$$

No universal index

GETTING MORE EFFICIENT

1st order Fermi acceleration: Shocks,



In a strong shock $\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} = 4$ for a fully ionized gas with $\gamma = \frac{5}{3}$

By mass conservation: $\rho_1 U_1 = \rho_2 U_2 \Rightarrow \frac{U_1}{U_2} = 4$

CYCLE: Particle first crosses upstream, starts with velocity U. In the rest frame of the upstream scatterers it has velocity $U + U_1$. After scattering flips direction and in the shock rest frame it has velocity $U + 2U_1$.

CYCLE CONTINUES:

Particle crosses downstream. In the rf of the scatterers it has velocity $U + 2U_1 - U_2$. After scattering it flips direction and in the shock rf it has velocity $U + 2U_1 - 2U_2$.

$$\frac{\Delta E}{E} = \frac{\frac{1}{2}m\left(U + 2(U_1 - U_2)\right)^2 - \frac{1}{2}mU^2}{\frac{1}{2}mU^2} = \frac{U^2 + 4(U_1-U_2)^2 + 4(U_1-U_2)U - U^2}{U^2}$$

$$\frac{\Delta E}{E} \simeq \frac{4(U_1-U_2)\,U}{U^2} = \frac{4(U_1-U_2)}{U}$$

$\left\langle \frac{\Delta E}{E} \right\rangle$ per collision is $\frac{1}{2}\,\frac{4(U_1-U_2)}{U} = \frac{2(U_1-U_2)}{U}$

β: factor by which particle KE increases per scattering

$$\beta = + \frac{2(U_1-U_2)}{U}$$

P: The probability that the particle remains in the acceleration zone after one collision

After K collisions number of particles still accelerating

$$\left.\begin{aligned} N &= N_0 P^K \\ \text{Their energy} \quad E &= E_0 \beta^K \end{aligned}\right\} \quad \frac{\ln P}{\ln \beta} = \frac{\ln(N/N_0)}{\ln(E/E_0)} \Rightarrow$$

$$\frac{N}{N_0} = \left(\frac{E}{E_0}\right)^{\frac{\ln P}{\ln \beta}} \Rightarrow \frac{dN}{dE} \propto E^{\frac{\ln P}{\ln \beta} - 1}$$

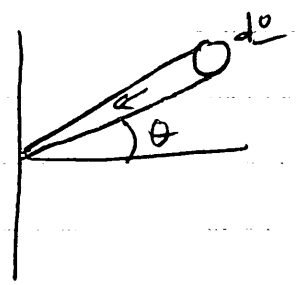
This is what escapes downstream

It is a power-law. But what is P?

Particles are isotropic in the downstream region and are advected away with a flux $v_2 N$, where N is the number density of accelerated particles.

Now N is the same upstream and downstream, since the accelerated particles do not "feel" the shock.

What is the particle flux upstream?



Assuming $v \simeq c$, the flux of particles crossing the shock per unit solid angle is

$$\left(\frac{dF}{d\Omega}\right)_{up} = \frac{Nc}{4\pi} \cos\theta \Rightarrow$$

$$F = \frac{Nc}{4\pi} \int_0^{2\pi}\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \, d\phi = \frac{Nc}{4}$$

The escape probability is the ratio of

$$\frac{\text{particles advected downstream}}{\text{particles crossing upstream}} = \frac{v_2 N}{\frac{Nc}{4}} = 4\frac{v_2}{c}$$

The probability that a particle remains in the acceleration region is:

$$P = 1 - 4\frac{v_2}{c}$$

Recall $\frac{dN}{d\epsilon} \propto \epsilon^{\frac{\ln P}{\ln \beta} - 1}$

$$P = 1 - \frac{4v_2}{c} \Rightarrow \ln(P) = \ln\left(1 - \frac{4v_2}{c}\right) \simeq -\frac{4v_2}{c}$$

$$\beta = \ln\left(1 + \frac{2\Delta u}{c}\right) \approx \frac{2\Delta u}{c}$$

So, $$\frac{dN}{d\epsilon} \propto \epsilon^{\frac{-\frac{4v_2}{c}}{2\frac{\Delta u}{c}} - 1} = \epsilon^{\frac{-4v_2}{2(v_1 - v_2)} - 1}$$

$$= \epsilon^{-\frac{2v_2}{3v_2} - 1} = \epsilon^{-5/3}$$

This is a "universal" index. If we had done the angular part of the DE right we would get the usual ϵ^{-2}.