PLASMA PHYSICS FOR ASTRONOMERS

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Preface

Plasma is a mixture of freely moving electric charges, its behaviour being determined by collective electromagnetic interactions. In the astronomical context, this means we are talking of electrons, protons and maybe positrons or heavy ions. The overwhelming majority of matter in the universe is ionised, astronomers study plasma objects when they are looking at stars, the interstellar medium or cosmic rays. The five introductory lectures aim to explain some of the basic concepts of plasma physics that are necessary for a understanding of astrophysical plasmas. We will however not spend time on lengthy and detailed derivations, but concentrate on the principles and the presentation of applicable results.

Acknowledgements:

The lectures use material from other lectures on plasma physics such as Skilling (Cambridge,1979), Fussmann (Berlin, 2001) and I am particularly grateful to Harald Lesch (Munich) for the use of illustrating material from his plasma physics lectures at the LMU in Munich.

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Definition of a Plasma

"…A plasma is a mixed gas or fluid of neutral and charged particles. Partially or fully ionized space plasmas have usually the same total number of positive (ions) and negative (electrons) charges and therefore behave quasineutral.

Space plasma particles are mostly free in the sense that their kinetic exceeds their potential energy, i.e. they are normally hot, T > 1000 K. Space plasmas have typically **vast dimensions**, such that the free paths of thermal particles are larger than the typical spatial scales --> they are collisionless…"(Lesch)

Plasmas differ by their chemical composition and the ionization degree of the ions or molecules (from different sources). Plasmas are mostly magnetized (internal and external magnetic fields).They occur in

- **lightening, discharge lamps**
- **Fusion reactors**
- **Solar interior and atmosphere**
- **Solar corona and wind (heliosphere)**
- **Planetary magnetospheres (plasma from solar wind)**
- **Planetary ionospheres (plasma from atmosphere)**
- **Coma and tail of a comet**
- **Dusty plasmas in planetary rings**
- **interstellar medium**
- **Nearly all matter considered by astrophysics**

Basic Equations

Plasma dynamics is governed by the interaction of the charged particles with the self-generated (by their motions through their charge and current densities) electromagnetic fields. These internal fields feed back onto the particles and make plasma physics difficult.

The motion of charged particles in space in strongly influenced by the self-generated electromagnetic fields, which evolve according to ***Ampere's and Faraday's*** (induction) laws:

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

where ε_0 and μ_0 are the vacuum dielectric constant and free-space magnetic permeability, respectively. The charge density is ρ and the current density **j**.

The electric field obeys ***Gauss*** law (Poisson's equation) and the magnetic field is always free of divergence, i.e. we have:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad\qquad \nabla \cdot \vec{B} = 0$$

In order to study the transport of plasma and magnetic field lines quantitatively, let us consider the fundamental induction equation, i.e. ***Faraday's law*** in combination with the simple phenomenological ***Ohm's law***, relating the electric field in the plasma frame with its current:

$$\vec{j} = \sigma_0 (\vec{E} + \vec{v} \times \vec{B})$$

The motion of charged particles in space is determined by the electrostatic ***Coulomb*** force and magnetic ***Lorentz*** force:

$$m \frac{d\vec{v}}{dt} = q_e (\vec{E} + \vec{v} \times \vec{B})$$

where q is the charge and **v** the velocity of any charged particle. If we deal with electrons and various ionic species (index, s), the ***charge and current densities*** are obtained, respectively, by summation over all kinds of species as follows:

$$\rho = \sum_s q_s n_s \qquad\qquad \vec{j} = \sum_s q_s n_s \vec{v}_s$$

which together obey the ***continuity equation***, because the number of charges is conserved, i.e. we have:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

Lorentz transformation of the electromagnetic field

Let S be an inertial frame of reference and S' be another frame moving relative to S at constant velovity **v**. Then the electromagnetic fields in both frames are connected to each other by the ***Lorentz transformation***:

$$\begin{aligned} E'_x &= E_x & \qquad B'_x &= B_x \\ E'_x &= \gamma (E_y - vB_z) & \qquad B'_y &= \gamma (B_y + \frac{v}{c^2} E_z) \\ E'_x &= \gamma (E_z + vB_x) & \qquad B'_z &= \gamma (B_z - \frac{v}{c^2} E_y) \end{aligned}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the ***Lorentz factor*** and c the speed of light. In the non-relativistic case, v << c, we have $\gamma = 1$, and thus **B'≈ B**. The magnetic field remains to lowest order unchanged in frame transformations. However, the electric field obeys, **E' ≈ E + v × B**. A space plasma is usually a very good conductor, and thus we have, **E' = 0**, and the result, **E ≈ - v × B**, which is called the ***convection electric field***.

Characteristic Plasma Parameters

SI units are used throughout and the notation follows that of NRL Plasma Formulary 1994 in most cases. All quantities are referred to the rest-frame of the plasma.

Particle density **n** ranges from 10^6 m^{-3} (ISM) to more than 10^{31} m^{-3} (stellar interior).
We use n_e, n_p etc. to designate densities of particle families.

Charge density $\rho = q_e \cdot (n_p - n_e)$ is distinct from the particle density **n**. If $\rho = 0$, then we speak of neutral plasma. However, not all astrophysical plasma is neutral (i.e pulsar magnetosphere).

Quantities derived from density:

Mean interparticle distance $\lambda_n = n_e^{-\frac{1}{3}}$

Plasma frequency (Langmuir frequency):

Assume that only electrons can move across an infinitely heavy background with $n_p = n_e$ and that charges are perturbed to move with a small velocity **u** away from their equilibrium position. Their eqn. of motion

$$m \cdot \frac{d}{dt} u = q_e \cdot E$$

describes the reaction of the particles to the electric field arising from the perturbation via Poisson's eqn.

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

Multiplying both sides of the eqn. of motion with q_e n_e and taking the divergence yields

$$m_e \nabla \cdot \frac{d}{dt}(q_e n_e u) = q_e \nabla E = \frac{q_e^2 n_e}{\varepsilon_0} \rho$$

Eliminating j using the continuity eqn. $\nabla j = \nabla(q_e n_e u) = -\frac{d}{dt}\rho$ gives the equation for harmonic oscillations in the charge density ρ:

$$\ddot{\rho} + \frac{q_e^2 \cdot n_e \cdot \rho}{m \cdot \varepsilon_0} = 0$$

with a characteristic frequency given by

$$\omega_{pl} = \sqrt{\frac{q_e^2 n}{m \cdot \varepsilon_0}}$$

Simple estimates are $\nu_{pl} = \frac{\omega_{pl}}{2\pi} = 8.98 \cdot \sqrt{n_e} \cdot \mathrm{Hz}$ (n_e in m^{-3}) and $\nu_{pl} = \frac{\omega_{pl}}{2\pi} = 8.98 \cdot \sqrt{n_e} \cdot \mathrm{kHz}$ (for n_e in cm^{-3}).

These charge density waves are the simplest plasma waves and are also called Langmuir waves.

The plasma frequency is referred to the rest frame of the plasma. If the plasma is streaming with a velocity $v_s=\beta_s \cdot c$ and a Lorentz factor of $\gamma_s = \sqrt{1-\beta_s}$ then the density n' in the observer frame is given by the transformation $n'=n/\gamma_s$. Hence in the observer frame the plasma frequency of *a streaming plasma* is given by

$$\omega_{pl} = \sqrt{\frac{q_e^2 n_0}{m \cdot \gamma_s \cdot \varepsilon_0}}$$

with n_0 being the density measured in the co-moving frame.
It is therefore higher that the rest frame frequency! This is important when i.e. the density of a relativistic streaming plasma is estimated from the observation of plasma related spectral features. One should also note, that a relativistic streaming plasma is not necessarily a ‘relativistic plasma’. A plasma is called relativistic, when its temperature is high enough, for the particle to have relativistic velocities in the rest frame of the plasma. Melrose & Gedalin (ApJ 521:351-361, 1999) discuss the difference and provide exact formulae for such cases.

Temperature **T** is often expressed as average kinetic energy using eV as a unit. ($1K = 8.61\cdot 10^{-5}$eV).

The thermal velocity of particles is then $v_{th} = \sqrt{\frac{k\cdot T_e}{m_e}}$.

An **ideal plasma** is characterized by **<$\Phi_{Coulomb}$>** $\leq$ **E_{th}** or $\frac{3}{2}\cdot k\cdot T > \frac{q_e^2}{4\cdot\pi\cdot\varepsilon_0}\cdot n^{\frac{1}{3}}$

which can be translated into $T > 1.1\cdot 10^{-5}\cdot n_e^{\frac{1}{3}}$ or $T_{eV} > 9.6\cdot 10^{-10}\cdot n_e^{\frac{1}{3}}$

High densities require consideration of quantum mechanical effects, important when the thermal energy 3/2 kT < E_F the Fermi energy $E_F = \frac{h^2}{(2\cdot\pi)^2\cdot 2\cdot m_e}\cdot\left(3\cdot\pi^2\right)^{\frac{2}{3}}\cdot n_e^{\frac{2}{3}}$

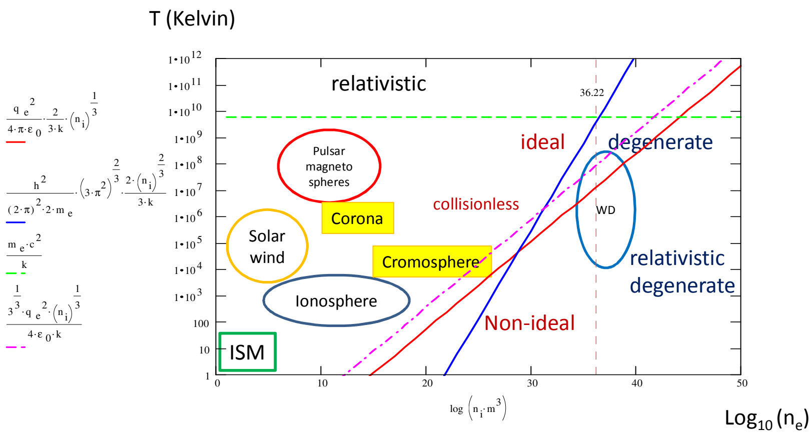
or $T < 2.8\cdot 10^{-15}\cdot n_e^{\frac{2}{3}}$ and $T_{eV} < 2.4\cdot 10^{-19}\cdot n_e^{\frac{2}{3}}$ (degeneracy limit)

For $T = 4.4\cdot 10^4$ and $n = \frac{8}{9}\cdot\frac{q_e^6}{\pi\cdot\varepsilon_0^3\cdot h^6}\cdot m_e^3 = 6.157\cdot 10^{28}\ \cdot m^{-3}$ we have a degenerate ideal plasma at a density of 0.103 of water if it is made of ionized hydrogen. The pressure will be about $5.7\cdot 10^9$ x atmospheric pressure,- conditions in stellar interiors.

Plasma is relativistic for $\frac{3}{2}\cdot k\cdot T > m_e\cdot c^2$ or $4\cdot 10^9$ K (= 341 eV).

It is relativistically degenerate for $E_F < m_e\cdot c^2$ or $n_e > \frac{16}{3}\cdot\frac{\sqrt{2}}{h^3}\cdot\pi\cdot m_e^3\cdot c^3 = 1.659\cdot 10^{36}\ \cdot m^{-3}$

n – T diagram for electron plasma



Three important length parameters depend on plasma density and temperature:

The ***Landau length*****,** given by

$$\lambda_L = \frac{q_e^2}{4\cdot\pi\cdot\varepsilon_0\cdot k\cdot T_e}$$

(or $\lambda_L=1.67\cdot 10^{-5} m\cdot T^{-1}$) is the distance at which the charge potential energy equals its thermal energy. It is the typical minimum distance of colliding charges. Electrons and ions can recombine when $\lambda_L>\lambda_n$.

The cross section for close collisions is of the order of $\pi\lambda_L^2$, hence the *mean free path* or *collision length* of particles will be

$$\lambda_{coll} = \frac{1}{\pi\cdot\lambda_L^2\cdot n_e} = \frac{16\cdot\pi\cdot\varepsilon_0^2\cdot k^2\cdot T_e^2}{q_e^4\cdot n_e} \quad \text{or} \quad \lambda_{coll} = 1.14\cdot 10^9\cdot\frac{T_e^2}{n_e}\cdot m$$

The collision frequency is the ratio of the mean particle velocity and the mean free path:

$$\nu_{coll} = \frac{\sqrt{3}\cdot q_e^4\cdot n_e}{16\cdot\pi\cdot\varepsilon_0^2\cdot m_e^{\frac{1}{2}}\cdot\left(k\cdot T_e\right)^{\frac{3}{2}}} \quad \text{or} \quad \nu_{coll} = 5.9\cdot 10^{-6}\cdot\frac{n_e}{T_e^{\frac{3}{2}}}\cdot s^{-1}$$

A plasma where the collision frequency is less than plasma frequency ($\nu_{coll} < \nu_{pe}$) is called a ***collisionless*** plasma.

Collisions determine the conductivity of a plasma in the absence of magnetic fields. *The* ***conductivity*** is then given by

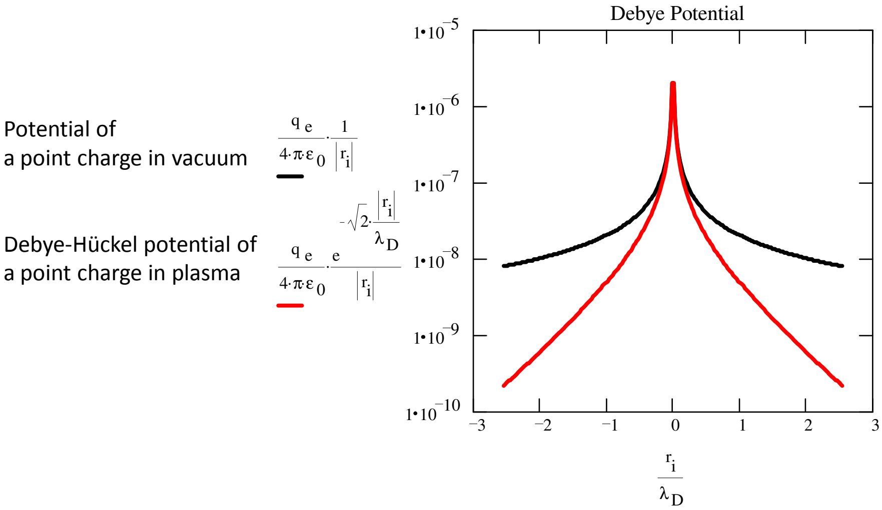
$$\sigma_{coll} = \frac{q_e^2\cdot n_e}{m_e\cdot\nu_{coll}}$$

Its inverse is the *resistivity (specific resistance)*:

$$\eta_{coll} = \frac{\sqrt{3}\cdot q_e^2\cdot m_e^{\frac{1}{2}}}{16\cdot\pi\cdot\varepsilon_0^2\cdot\left(k\cdot T_e\right)^{\frac{3}{2}}} \quad \text{or} \quad \eta_{coll} = \frac{210}{T_e^{\frac{3}{2}}}\cdot\Omega\cdot m^{-1}$$

One should note, that the collisional resistivity does not depend on the charge density, but only on the temperature of the electron gas as the higher charge density provides more charge carriers but at the same time increases the collision rate in the same way. A 1 keV plasma (10^7 K) has a conductivity of $2\cdot 10^7$ $(\Omega m)^{-1}$ which is similar to copper ($\sigma=5.9\cdot 10^7$ $(\Omega m)^{-1}$).

A charge in a plasma will attract charges of opposite polarity. Because of that, its attraction will have a finite range in a plasma (*screening length or Debye length*). The presence of neutralizing charges modifies the effective potential of a point charge



A simplified 1-D argument will illustrate the magnitude of the screening length for a particle displaced by a small distance x from its equilibrium position.

Integration of the Poisson eqn. $\frac{\partial E(x)}{\partial x} = -\frac{q_e}{\varepsilon_0} n_e$ over x yields the electric field in the x-direction

$E(x) = -\frac{q_e}{\varepsilon_0} n_e x$. Hence moving the charge by a distance λ_D against the background of neutralizing charges will change its potential energy by $\int_0^{\lambda_D} q_e \cdot E(x)\, dx = \frac{1}{2}\cdot\frac{n_e}{\varepsilon_0}\cdot\lambda_D^2$. Setting this energy equal to the thermal energy per degree of freedom ½kT will yield the expression for the Debye screening length

$$\lambda_D = \sqrt{\frac{\varepsilon_0 \cdot k \cdot T_e}{q_e^2 \cdot n_e}} \quad \text{or } \lambda_D = 6.9\cdot\sqrt{\frac{T_e}{n_e}}\cdot m$$ in SI units. For Te in eV the factor is 7437.

One may note, that $\omega_{pl} = \frac{v_{th}}{\lambda_D}$, the plasma frequency being the ratio of the typical thermal speed and the Debye length. When the average particle distance λ_n exceeds the Debye length, then there will not be sufficient charges $N_D = n_e \cdot \frac{4}{3}\cdot\pi\cdot\left(\frac{\lambda_D}{2}\right)^3$ within the Debye volume to screen individual charges. In this case one is dealing with a cloud of isolated charges, which is not a plasma but a hot gas, consisting of charged particles without collective effect.

The condition $N_D>1$ can be reformulated into a condition for the plasma temperature:

$T>n_e^{\frac{1}{3}} \cdot \left(\frac{3}{4\cdot\pi}\right)^{\frac{2}{3}} \cdot \left(\frac{q_e^2}{\varepsilon_0 \cdot k}\right)$ It corresponds to the ideal plasma condition discussed above.

The plasma parameter

$$\Lambda = \frac{1}{n_e \cdot \lambda_D^3} = 4\cdot\pi\cdot\frac{\lambda_L}{\lambda_D}$$

is smaller than unity for any true plasma and individual particle interactions are happening mainly within the Debye sphere. Its logarithm $-\ln(\Lambda)$ is called the Coulomb logarithm. It is a measure of the of the average scattering deflection.

Plasma frequencies for different media							
medium	density (m^{-3})	T (K)	ν_{pl} (MHz)	λ_n	λ_D	λ_{coll}	λ_L
ISM	1,00E+06	1,00E+04	8,98E-03	0,01	6,90E-01	1,14E+11	1,67E-09
upper Ionosphere	1,00E+08	1,00E+05	8,98E-02	0,002154	2,18E-01	1,14E+11	1,67E-10
lower ionosphere(F)	1,00E+12	1,00E+03	8,98E+00	0,0001	2,18E-04	1,14E+03	1,67E-08
solar corona	1,00E+14	1,00E+06	8,98E+01	2,15E-05	6,90E-04	1,14E+07	1,67E-11
Hg	1,00E+29	1,00E+02	2,84E+09	2,15E-10	2,18E-13	1,14E-16	1,67E-07
stellar interior	1,00E+31	1,00E+08	2,84E+10	4,64E-11	2,18E-11	1,14E-06	1,67E-13

Magnetic Fields and Plasma:

Gyro-, (Larmor, Cyclotron) Frequency:

The Lorentz force $m\frac{d\vec{v}}{dt} = q_e\,\vec{v}\times\vec{B}$ causes particles to move in circles (gyrate) perpendicular to the magnetic field with a frequency (**gyro or cyclotron frequency**) given by $\omega_c = \frac{|q_e|\cdot B}{m}$ which does not depend on particle energy (in the non relativistic case). For electrons we get $\frac{\omega_{ce}}{2\cdot\pi} = 2.8\cdot MHz\cdot\left(\frac{B}{Gauss}\right)$ and for protons it is $\frac{\omega_{cp}}{2\cdot\pi} = 1.52\cdot kHz\cdot\left(\frac{B}{Gauss}\right)$

The **gyro-radius** is given by $r = \frac{v_{th}}{\omega_c}$, which for electrons is $r_{ge} = \frac{\sqrt{m_e\cdot k\cdot T}}{q_e\cdot B} = 2.2\cdot 10^{-4}\cdot m\cdot\frac{\sqrt{T}}{B}$ and for protons it is given as $r_{gp} = \frac{\sqrt{m_p\cdot k\cdot T}}{q_e\cdot B} = 9.5\cdot 10^{-3}\cdot m\cdot\frac{\sqrt{T}}{B}$ (here B is given in Gauss for convenience).

A quantum mechanical treatment (Landau levels) of the particle motion will be required when $r_g < \lambda_{deBroglie} = \frac{h}{2 \cdot \pi \cdot \sqrt{m \cdot k \cdot T}}$. For electrons this happens when $B > \frac{2 \cdot \pi \cdot m_e \cdot k \cdot T}{q_e \cdot h} = 0.744 \cdot \mathrm{Tesla} \cdot T$ and for protons $B > \frac{2 \cdot \pi \cdot m_p \cdot k \cdot T}{q_e \cdot h} = 1367 \cdot \mathrm{Tesla} \cdot T$. In the pulsar magnetosphere we have B=10^8 Tesla at the surface and an electron temperature of about 10^6 K. That means, that q.m. will be important out to 5.2 pulsar radii. But for recycled (ms) pulsars with surface fields of B=10^4 Tesla this is not the case.

Charges moving in static fields

- **Gyromotion of ions and electrons**
- **Drifts in electric fields**
- **Inhomogeneous magnetic fields**
- **Magnetic and general drift motions**
- **Trapped magnetospheric particles**
- **Motions in a magnetic dipole field –planetary radiation belts**

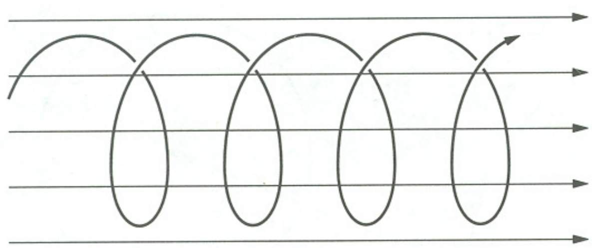
Case E||B: i.e. E=(0,0,E), B=(0,0,B)

The Lorentz force $m\frac{d\vec{v}}{dt} = q_e(\vec{E} + \vec{v} \times \vec{B})$ causes particles to move in circles (gyrate) perpendicular to the magnetic field and an acceleration in the direction of E.

We split v into ‖ and ⊥ components with $v_{\parallel}=v_z$ and $v_{\perp}=(v_x^2+v_y^2)^{1/2}$ and write

$$\begin{aligned} \dot{v}_x &= \frac{q}{m}Bv_y \\ \dot{v}_y &= -\frac{q}{m}Bv_x \\ \dot{v}_z &= \frac{q}{m}E \end{aligned} \quad \Rightarrow \quad \begin{aligned} \ddot{v}_x &= -\frac{q^2}{m^2}B^2 v_x = \omega_g{}^2 v_x \\ \ddot{v}_y &= -\frac{q^2}{m^2}B^2 v_y = \omega_g{}^2 v_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{v_\perp}{\omega_g}\sin(\omega_g t) \\ y &= \frac{v_\perp}{\omega_g}\cos(\omega_g t) \\ z &= \frac{q}{2m}E \cdot t^2 + v_{\parallel} t \end{aligned}$$

The result is a helicoidal charge motion in a uniform magnetic field



$$r_g = \frac{v_\perp}{|\omega_g|} = \frac{m v_\perp}{|q|B} \qquad \alpha = \arctan\left(\frac{v_\perp}{v_\parallel}\right)$$

If one includes a constant speed parallel to the field, the particle motion is three-dimensional and looks like a *helix*. The *pitch angle* α of the helix or particle velocity with respect to the field depends on the ratio of perpendicular to parallel velocity components. (Lesch)

positive charges move on right handed spirals, negative charges on left handed spirals.

Case E⊥B:

A perpendicular electric field component (E,0,0) leads to the famous $\boldsymbol{E} \times \boldsymbol{B}$ drift:

$$\dot{v}_x = \omega_g v_y + \frac{q}{m} E_x \qquad \ddot{v}_x = -\omega_g{}^2 v_x$$
$$\dot{v}_y = -\omega_g v_x \quad \rightarrow \quad \ddot{v}_x = -\omega_g{}^2 (v_y + \frac{E_x}{B})$$

Substituting $v_x' = v_x \quad v_y' = v_y + \frac{E_x \cdot B_z}{B^2}$ we get the same system of equations as above. Therefore the solution for y is $v_y = v_\perp \cos(\omega_g t) + \frac{E_x \cdot B_z}{B^2}$.

For general $E=(E_x,E_y,E_z)$ and $B=(0,0,B_z)$ we have

$$v_x = v_\perp \cos(\omega_g t) + \frac{-E_y \cdot B_z}{B^2} \qquad v_y = v_\perp \cos(\omega_g t) + \frac{E_x \cdot B_z}{B^2}$$

where the drift velovitiy is $v_D = \frac{E \times B_z}{B^2}$ and for a general B that will become $\vec{v}_D = \frac{\vec{E} \times \vec{B}}{B^2}$, the $\vec{E} \times \vec{B}$ drift.

Note: E x B drift is independent of charge sign, but this is not the case for drifts caused by other forces.

In the case of a general force $m\frac{d\vec{v}}{dt} = F + q_e \vec{v} \times \vec{B}$ we can transform to a coordinate system moving with v=v'+v_D with $v_D = \frac{\vec{F} \times \vec{B}}{qB^2}$ so that again $m\frac{d\vec{v'}}{dt} = q_e \vec{v'} \times \vec{B'}$

As a result we have purely gyrating motion around a moving centre => guided centre motion.

Example:

At the pulsar surface we have $B_0 \approx 10^8$T, $E_0 \approx (\Omega x B_0) \cdot r_{ns} = 10^{13}$ V/m → $v_D \approx 10^5$ m/s. For a polar cap size of say r_{cap}=100m we get a drift period of about 3 ms. At the same time one can calculate the gravity drifts for e (10^{-7} m/s) and p (10^{-4} m/s). Heavy particles show much stronger drifts and for gravity the sign is important. The drift current density is given by $j = n_e \cdot \left(\frac{m_i}{Z_i} + m_e\right) \cdot \frac{g \times B}{B^2}$ or about 10^{-6} Am^{-2} at the surface. Gravity does not play a role, not even for drift effects.

For this approximation to be valid, B' should not change much over the time of one gyration:

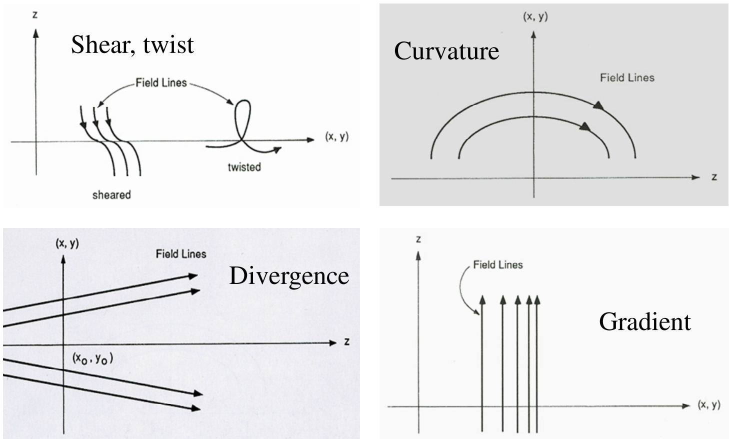
$$\frac{1}{\omega_g}\frac{\partial B}{\partial t} << 1 \text{ and } \frac{|(\nabla \cdot v)B|}{\omega_g |B|} << 1$$

Inhomogeneous Fields

Magnetic fields are the consequence of current distributions:

$\nabla\times\overline{B}=\frac{1}{c^2}\dot{\vec{E}}+\mu_0\vec{j}$ (Ampere law) and will not be homogeneous or curvature free on all scales.

Nonuniform magnetic fields in space

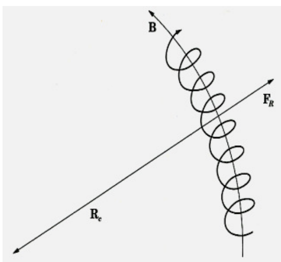


(Lesch)

Stationary fields have at least $\dot{\vec{E}}=0$ and outside the current carrying region j=0, hence $\nabla\times B=0$.

We use the vector identity (NRL formula 11, pg. 4) $(B\cdot\nabla)B=B\cdot(\nabla B)-B\times\nabla\times B$ to get

$\frac{\nabla B}{|B|}=\frac{(B\cdot\nabla)B}{B^2}=-\frac{\vec{e}_R}{R}$ where R is the local radius of curvature of B and $\mathbf{e}_R$ the unit vector perpendicular to B but in the curvature plane.



Centrifugal forces F_R are in the direction of $\mathbf{e}_R$.

Particles following stationary field lines may experience a gradient B (1) and an inertial (centrifugal) force (2).

(1) gyrating particles have a magnetic moment

$$\mu=j_{gyr}\cdot\pi\cdot r_g{}^2=\frac{q_e\omega_g}{2\pi}\cdot\pi\frac{v_\perp{}^2}{\omega_g{}^2}=\frac{q_e}{2}v_\perp{}^2\frac{m}{q_e|B|}=\frac{E_{kin\perp}}{|B|}$$

its direction is opposite to B → $\mu = -\frac{m}{2} v_\perp{}^2 \frac{\vec{B}}{|B|^2}$. As a general rule, external fields result in electrical and magnetic polarization of plasma which leads to weakening of external fields. The magnetic moment couples to the gradient of B: $F_\nabla = \mu \nabla |B|$ and with the help of our F x B drift formula we get $v_\nabla = \vec{\mu} \cdot \frac{\nabla B \times \vec{B}}{q_e B^2} = -\frac{m}{2} v_\perp{}^2 \cdot \frac{\nabla B \times \vec{B}}{q_e B^3}$. The drift direction is ⊥ to B and to its gradient.

(2) the inertial force is caused by the particle following a curved trajectory:

$$F_{curv} = m v_\parallel{}^2 \frac{\vec{e}_R}{R} = -m v_\parallel{}^2 \frac{(B \cdot \nabla)B)}{B^2}$$

Hence the drift velocity will be given by

$$v_D = \frac{(F_\nabla + F_{curv}) \times \vec{B}}{B^2} = -\frac{m}{2 q_e B^2} (v_\perp{}^2 \cdot \frac{\nabla B \times \vec{B}}{q_e |B|} + 2 v_\parallel{}^2 \cdot \frac{(B \cdot \nabla)B}{B^2} \times \vec{B})$$

The drift forces particles away from regions of high curvature and gradients. The drift is charge specific and can create net currents.

Summary of guiding center drifts

E×B Drift:	$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$	
Polarization Drift:	$\mathbf{v}_P = \frac{1}{\omega_g B} \frac{d\mathbf{E}_\perp}{dt}$	$\mathbf{j}_P = \frac{n_e(m_i + m_e)}{B^2} \frac{d\mathbf{E}_\perp}{dt}$
Gradient Drift:	$\mathbf{v}_\nabla = \frac{m v_\perp^2}{2q B^3} (\mathbf{B} \times \nabla B)$	$\mathbf{j}_\nabla = \frac{n_e(\mu_i + \mu_e)}{B^2} (\mathbf{B} \times \nabla B)$
Curvature Drift:	$\mathbf{v}_R = \frac{m v_\parallel^2}{q R_c^2 B^2} (\mathbf{R}_c \times \mathbf{B})$	$\mathbf{j}_R = \frac{2n_e(W_{i\parallel} + W_{e\parallel})}{R_c^2 B^2} (\mathbf{R}_c \times \mathbf{B})$

Associated with all these drift are corresponding drift currents.

(Lesch)

Adiabatic Invariants

Adiabatic invariants of motion

In classical Hamiltonian mechanics the action integral

$$J_i = \oint p_i dq_i$$

is an invariant of motion for any change that is slow as compared to the oscillation frequency associated with that motion. Three invariants related to:

- *Gyromotion* about the local field $\Phi_\mu = \frac{2\pi m}{q^2}\mu = \text{const}$
- *Bounce motion* between mirror points $J = \oint m v_\parallel ds$
- *Drift motion* in azimuthal direction $\Phi = \frac{2\pi m}{q^2} M = \text{const}$

Magnetic flux, $\Phi = B\pi r_g^2$, through surface encircled by gyro orbit is constant.

(Lesch)

1. First adiabatic invariant

For slowly varying fields ($R >> r_g$, $\frac{\dot{B}}{B} << \omega_c$) the magnetic moment $\mu = \frac{m}{2}\frac{v_\perp^2}{B} = \frac{m}{2} B \cdot r_g^2$ is conserved. But energy conservation means that $E_{kin} = \frac{m}{2}(v_\parallel^2 + v_\perp^2)$ is also constant. Particles move on a helical trajectory with a pitch angle given by $\sin^2\theta = \frac{v_\perp^2}{v_\parallel^2 + v_\perp^2}$ which means that $\frac{E_{kin}}{|B|} \cdot \sin^2\theta$ has to be a conserved quantity. Hence we have

$$\sin^2\theta = \frac{B}{B_0} \cdot \sin^2\theta_0 < 1$$

for any particle that has a pitch angle θ_0 at a given field B_0. The pitch angle will increase for increasing B, which means that $v_\perp$ increases at the expense of $v_\parallel$ and for $B > \frac{B_0}{\sin^2\theta_0}$ the parallel motion will cease.

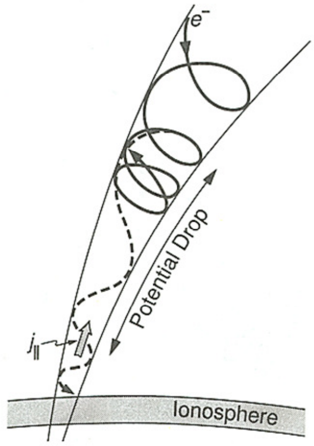
The first adiabatic invariant has important consequences:

1.1 Mirror confinement slow charges by converging B-fields is the result of the pitch angle limit above.

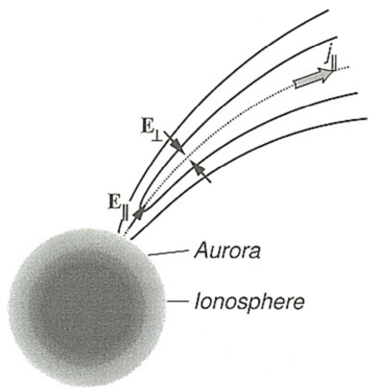
1.2 Fast charges, that have a pitch angle $\sin^2\theta_0 < \frac{B_{max}}{B_0}$ can leave the confinement which will lead to a non-thermal 'loss-cone' particle energy distribution.

1.3 Sideways compression of plasma by magnetic fields will convert gyro energy into parallel kinetic energy: particles heat up! However strong fields are required for strong effects.

Acceleration of auroral electrons



Mirror impedance due to a field-aligned electric potential drop

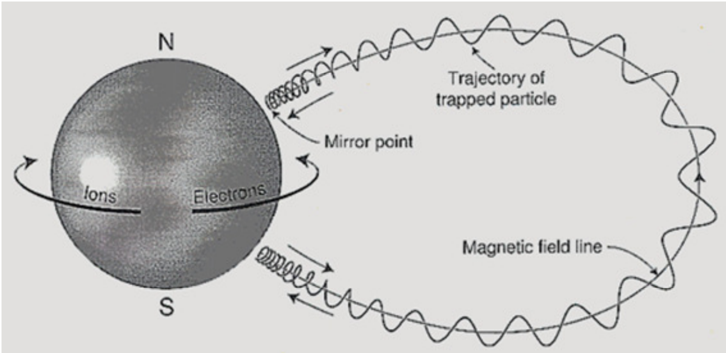


Acceleration of auroral electrons by a V-shaped electric potential

(Lesch)

Van Allen Belts

Trajectories of particles confined in a dipole field



A *dipole magnetic field* has a field strength minumum at the equator and converging field lines at the polar regions (mirrors). Particles can be *trapped* in such a field. *They perform gyro, bounce and drift motions.*

(Lesch)

The outer belt extends 3-10 earth radii and consists of trapped electrons with energies ranging from 0.1 – 10 MeV. The gyro radii of electrons are below few 10 km and therefore small compared to the extent of the van Allen belt. The inner belt starts at a height of about 70 km and extends out to about 10000 km. Electrons and protons have energies in excess of 100keV and 50 MeV are sometimes reached. Particle fluxes above 500 keV are of the order of 10^6 – 10^{10} $cm^{-2}s^{-1}$.

For $B=10^{-6}$ Tesla (0.01 G) and 100 keV electrons we get $v = 1.5 \cdot 10^8$ m/s, $\omega_{ge}=1.5 \cdot 10^5$ rad/s (25 kHz) and $r_{ge} \approx 1$ km. Hence the first adiabatic invariant will be conserved as B doesn't change much over 40 microseconds or 40 km. Leakage will however occur at the poles and by pitch angle / loss-cone scattering.

Inhomogeneity and curvature will create charge dependent drifts which set up an equatorial current.

Take $R_{curv}=10 \cdot r_{earth} \approx 64000$km and the lowest kinetic energy of $E_{kin}=100$keV, then $v_D = \frac{2 \cdot E_{kin}}{q_e B^2} \cdot \frac{(B \cdot \nabla)B}{B^2} \times \vec{B}) \approx \frac{2 \cdot E_{kin}}{q_e B} \cdot \frac{1}{R_{curv}} \approx 3 \frac{km}{s}$ and with the particle flux of 10^6 $cm^{-2}s^{-1}$ we obtain a very small current density of 1.6 $\mu A \cdot m^{-2}$. However the total cross-section of the van Allen belt is huge $\approx 10^4$ km x 10^4 km = 10^{14} m^2 and consequently we are dealing with a huge total current of $\approx 10^8$ A. The charges in the van Allen Belts originate in the fluctuating solar wind and thus can fluctuate themselves by an order of magnitude or more which can induce low frequency currents on long distance electrical transmission lines. These can simply trip circuit breakers or even damage to transformers and other equipment, leading to power outages for large areas.

2. The second (longitudinal) adiabatic invariant relates to the periodic motion of particles in a magnetic confinement $J = \int_a^b v_{\|} ds$ where the integration is along the path of the charges from one reflection point *a* to the other *b* and back. As a consequence, particles are restricted to field lines where J is the same. In a field configuration that has cylindrical symmetry, the particle will be radially confined in a hollow cylinder, but will eventually return to the field lines they had started on. This requires B to change more slowly than transit times, which for the van Allen Belts are about $10 \cdot r_{earth}/v\| \approx 0.3$ s. But when the magnetic mirror changes (i.e. oscillates) at the transit frequency, particle heating can be the result. The Fermi acceleration of cosmic ray particles bouncing between converging magnetised interstellar plasma clouds is a mechanism explicable by adiabatic invariants of the trapped cosmic ray particle motion.

Plasma Waves

"…In a plasma there are many reasons for spatio-temporal variations (waves or more generally fluctuations): High temperature required for ionization (Φ_H = 13.6 eV ≈ 158000 K) implies fast *thermal particle motion*. As a consequence

➔ **microscopic fluctuating charge separations and currents ➔ fluctuating electromagnetic fields.**

There are also externally imposed disturbances which may propagate through the plasma and spread their energy in the whole plasma volume.

Plasma waves are not generated at random. To exist they must satisfy two conditions:

➔ **their amplitude must exceed the thermal noise level**

➔ **they must obey appropriate dynamic plasma equations**

There is a large variety of wave modes which can be excited in a plasma. The mode structure depends on the composition, boundary conditions and theoretical description of the plasma..." (Lesch)

We may represent any wave disturbance, $\boldsymbol{A}(\mathbf{x},t)$, by its Fourier components (with amplitude, $\boldsymbol{A}(\mathbf{k},\omega)$, wave vector **k**, and frequency, ω):

$$A(x,t) = A(\vec{k},\omega)\cdot e^{-i(\omega t-\vec{k}\cdot\vec{x})}$$

where $\left|\vec{k}\right| = \frac{2\pi}{\lambda}$

Phase velocity (wave front propagation): $v_p = \frac{\omega}{k}$

Group velocity (energy and information transport): $v_p = \frac{\partial\omega}{\partial k}$

The phase velocity is defined as the rate at which the *crests* of the waveforms propagate; that is, the rate at which the phase of the waveform is moving. The group velocity is the rate at which the *envelope* of the waveform is propagating; that is, the rate of variation of the amplitude of the waveform. Provided the waveform is not distorted significantly during propagation, it is the group speed that represents the rate at which information (and energy) may be transmitted by the wave (for example, the speed at which a pulse of light travels down an optical fibre) . See Feynman Lectures Vol. I pg. 48-6 for details.

The **refractive index**, *n*, of a medium is defined as the ratio of the speed, *c*, of a wave to the phase velocity v_p, of the wave in the medium in question:

$$n = \frac{c}{v_p} = \frac{ck}{\omega}$$

It is usually given the symbol *n*. In the case of light, it equals

$$n = \sqrt{\epsilon_r\mu_r},$$

where ϵ_r is the material's relative permittivity, and μ_r is its relative permeability. For most naturally occurring materials, μ_r is very close to 1, therefore *n* is approximately $e_r^{1/2}$. (Wikipedia)

n may be less than 1 for electromagnetic waves in plasmas as we will see later on.

Classification of Plasma waves

(from Fussman 2001)

$B_{ext}=0$

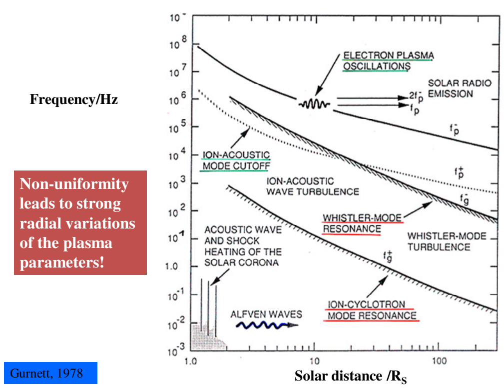
$k \perp E$	$k \parallel E$
transverse el.mag. waves	longitudinal electron-acoustic (Plasmon) waves Ion-acoustic waves

$B_{ext} \neq 0$

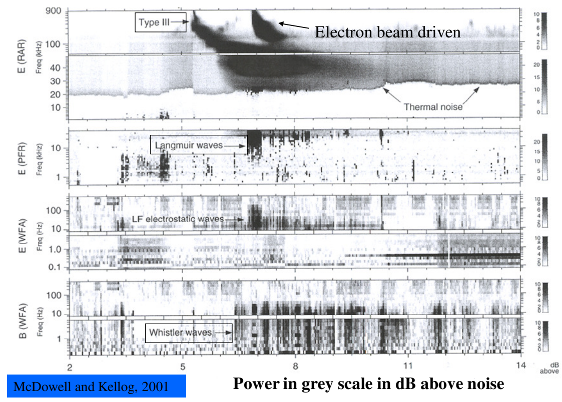
$k \parallel B$		$k \perp B$	
$E \parallel B$	$E \perp B$	$E \parallel B$	$E \perp B$
electron + ion acoustic	circ. polarized el. cycloton (Whistlers) ion cycl. waves Alfven torsion waves	O-mode transv. el.mag.	X-mode upper hybrid lower hybrid Alfven compression (magneto-acoustic) electrostatic ion cyclotron Bernstein waves

Many of these various wave modes are present in the solar wind

Plasma waves and frequencies in the solar wind



Ulysses wave data - day 73 in 1995



and have been detected by the Ulysses satellite experiment.

Example 1: transverse e.m. waves

Derivation of the dispersion relation for transverse e.m. waves (k⊥E) in a B=0, v_{th}=0 (cold) electron plasma, infinitely homogeneous and in equilibrium. Ion motion is neglected.

First order wave approximation: $A = A_0 + A_1(\omega, t)$ with <A>=0 → A_0=0

and with stationarity condition: $\frac{\partial}{\partial t} < A >= 0 \quad \rightarrow \quad \frac{\partial}{\partial t} A_0 = 0$.

Now use

(1) eqn. of motion: $m_e \frac{\partial}{\partial t} v_1 = q_e(E_1 + \underbrace{v_1 \times B_1}_{O(\varepsilon^2)\to 0})$ and by substituting $A_1 \rightarrow A_1 e^{i\omega t}$

we get the linearized first order equation of motion:

$$i\omega m_e \cdot v_1 = q_e E_1$$

(2) Ohm law: $j_1 = q_e n_e v_1 = \sigma \ (E_1 + \underbrace{v_1 \times B_1}_{O(\varepsilon^2)\to 0})$

which we can solve for the conductivity σ by eliminating E and v in (1) and (2):

$$\sigma = \frac{-iq_e{}^2 n_e}{\omega \cdot m_e}$$

The conductivity here is purely imaginary, leading only to frequency dependent phaseshifts, but no dissipation. The cold electron plasma is seen as a loss less dielectric medium.

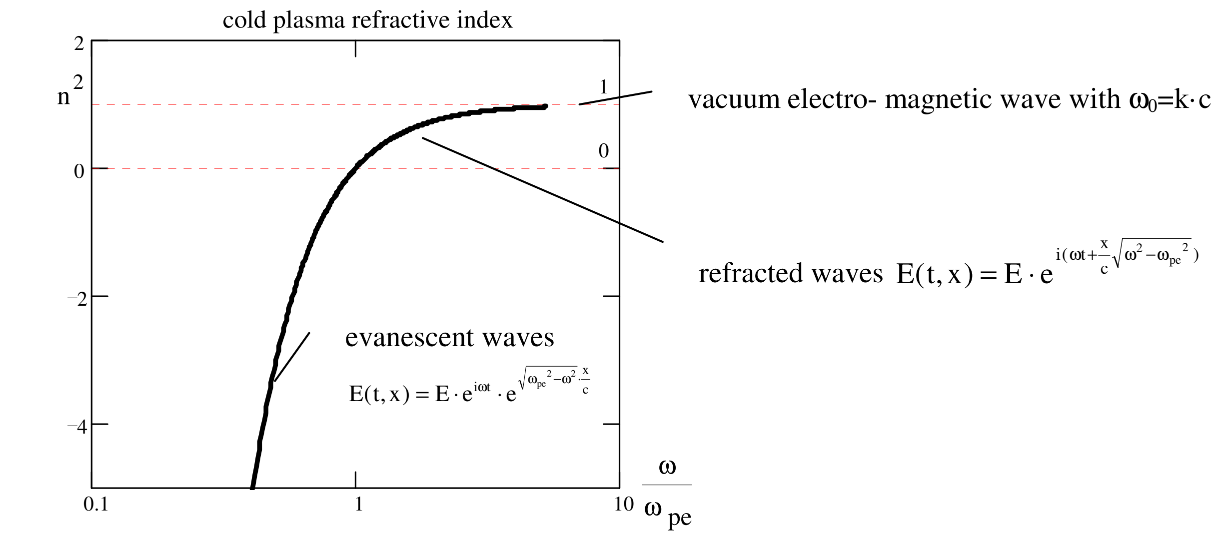
(3) Ampere law: $\nabla \times B = \mu_0(\varepsilon_0 \frac{\partial E_1}{\partial t} + \sigma E_1 + \underbrace{v_1 \times B_1}_{O(\varepsilon^2)\to 0})$ for which harmonic first order expansion

yields $= \mu_0\varepsilon_0(i\omega - \frac{i \cdot q_e{}^2 n_e}{\omega \cdot \varepsilon_0 \cdot m_e}) \cdot E_1 \quad = i\omega \cdot \mu_0 \cdot \varepsilon_0 \cdot (1 - \frac{q_e{}^2 n_e}{\omega^2 \cdot \varepsilon_0 \cdot m_e}) \cdot E_1$

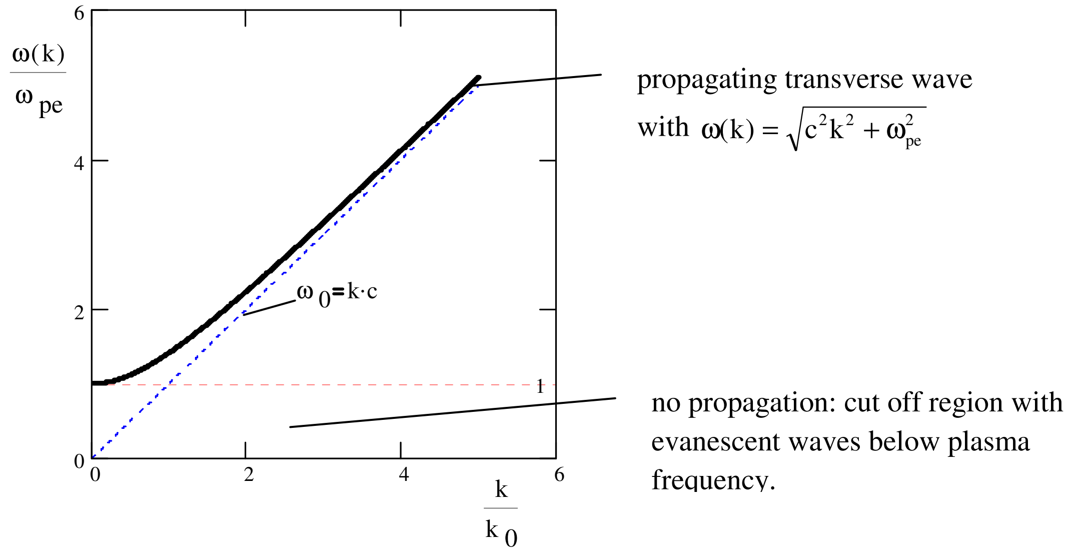
We identify the factor $(1 - \frac{q_e{}^2 n_e}{\omega^2 \cdot \varepsilon_0 \cdot m_e})$ as the relative dielectric constant $\boldsymbol{\varepsilon_r}$ and with the definition of the plasma frequency $\omega_{pe}^2 = \frac{i \cdot q_e{}^2 n_e}{\omega^2 \cdot \varepsilon_0 \cdot m_e}$ we finaly obtain the dispersion relation for transverse waves in a cold electron plasma:

$$n^2 = \varepsilon_r = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{pe}{}^2}{\omega^2}$$

The generic wave is written as $E \cdot e^{i(\omega t + k \cdot x)}$ and for $n^2 < 0$ or $\omega < \omega_{pe}$ we find that waves have imaginary k leading to a description of decay of waves attenuated with distance x: $E(t,x) = E \cdot e^{i\omega t} \cdot e^{\sqrt{\omega_{pe}{}^2 - \omega^2} \cdot \frac{x}{c}}$, these are called *evanescent waves*. Their absorption length is $\tau = \frac{c}{\sqrt{\omega_{pe}{}^2 - \omega^2}}$



we can also plot $\omega(k) = \sqrt{c^2k^2 + \omega_{pe}^2}$ with $k_0 := \frac{\omega_p}{c}$



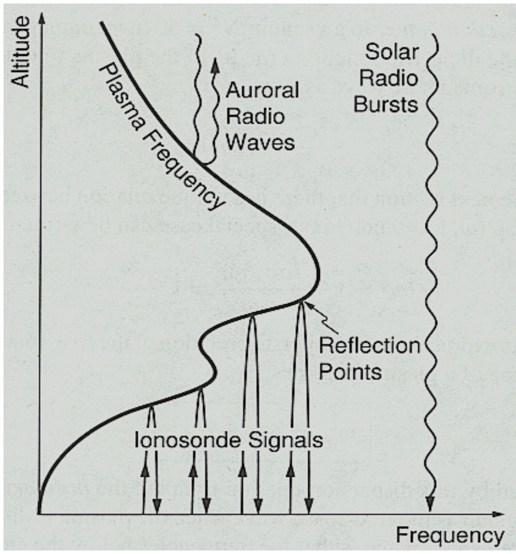
The phase velocity is of the transverse waves is $v_{ph} = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}}$ and the group velocity is given by

$$v_{gr} = \frac{\partial\omega}{\partial k} = \frac{k\ c^2}{\sqrt{k^2c^2 + \omega_{pe}^2}} = c \cdot \sqrt{1 - \frac{\omega_{pe}^2}{\omega^2}}.$$

For large ω, the plasma waves will be indistiguishable from ordinary electro-magnetic waves. But the group velocity becomes an imaginary quantity for $\omega < \omega_{pe}$ which implies that evanescent waves will not transmit information as their modulation will be strongly attenuated and dispersed. Waves with $\omega < \omega_{pe}$ will suffer total reflection by the plasma at its outer boundary.

Consequences:

1. reflections of short-wave radio transmissions in the earth's ionosphere which have enabled global radio communications using comparatively small transmitters. Terrestrial Radio astronomy is only feasible on frequencies above the short wave range (20-30 MHz), but the earth transmits radio waves to outer space on frequencies corresponding to the ionospheric plasma frequency from the aurora .



(Lesch)

2. reflectivity / lustre of metals: electron densities in the conduction band of metals are of the order of $10^{29}m^{-3}$, which results in plasma frequencies of $\approx 3\cdot 10^{15}$ Hz or wavelengths of λ=100 nm which in the high ultraviolet region. Metals will reflect e.m. waves (including light) below that frequency. Non-conductors will not do so and will therefore be either transparent or absorbent.

3. interstellar dispersion: pulses and transmitted power travels with the group velocity, hence the arrival time t of a signal transmitted at a frequency ν over a distance d in the interstellar medium will be

$$t(\nu) = \frac{d}{c\sqrt{1-\frac{\omega_{pe}^2}{\omega^2}}} \approx \frac{d}{c}\left(1+\frac{1}{2}\left(\frac{\nu_{pe}}{\nu}\right)^2 + O\left(\left(\frac{\nu_{pe}}{\nu}\right)^4\right)\right),-$$

we are allowed to neglect higher orders in $\frac{\nu_{pe}}{\nu}$ as $\nu_{pe} \approx 1\text{kHz}$ for the interstellar medium.

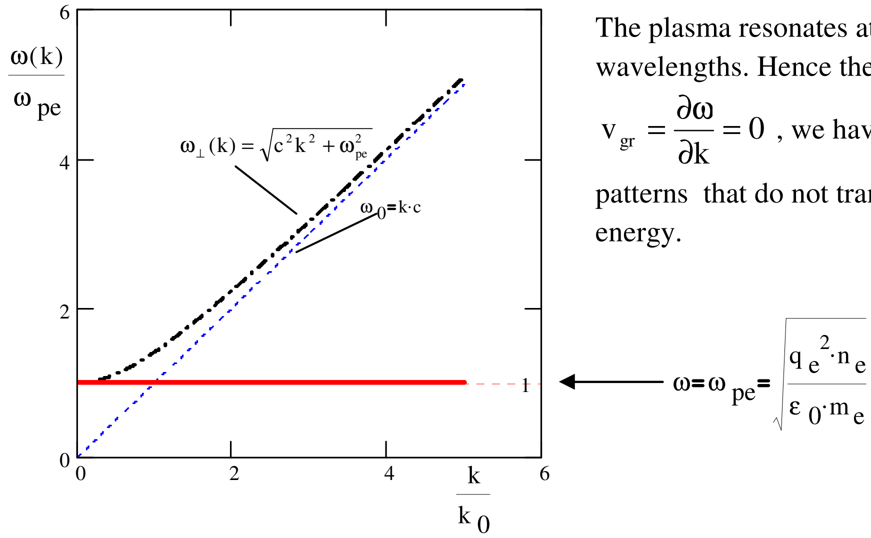
The dispersion delay $\delta t(\nu) = t(\nu) - t(\infty) = \frac{d\cdot q_e^2 \cdot <n_e>}{2\cdot(2\pi)^2\cdot m_e c}\frac{1}{\nu^2}$ is directly proportional to the average electron density <n> and inversely proportional to the square of the frequency, high frequency signals arrive earlier that low frequency signals, an effect much in evidence in pulsar observations. We can express the average as an integral over the line of sight:

$$<n_e> = \frac{1}{d}\int_0^d n_e(s)ds \quad \rightarrow \quad \delta t(\nu) = \frac{q_e^2}{8\pi^2\varepsilon_0 m_e c}\underbrace{\int_0^d n_e(s)ds}_{DM}\cdot\frac{1}{\nu^2} = 4.15\cdot 10^{15}s^{-1}\cdot\frac{DM}{\nu^2}$$

and get the canonical expression for the interstellar dispersion delay shown above.

Example 2: longitudinal electrostatic waves k||E in cold plasma

These are nothing else then the density waves at the plasma frequency described in the first lecture!



The plasma resonates at ω_{pe}, for all wavelengths. Hence the group velocity $v_{gr} = \frac{\partial \omega}{\partial k} = 0$, we have standing wave patterns that do not transmit information or energy.

The phase velocity $v_{ph} = \frac{\delta(\omega - \omega_{pe})}{k} = \begin{cases} \infty & \omega = \omega_{pe} \\ 0 & \omega \neq \omega_{pe} \end{cases}$ and consequently the refractive index vanishes too (n_{ref}=0). The oscillations happen with infinite phase velocity ('superluminal') and can have arbitrary wavelengths. It becomes clear, that this is a somewhat extreme idealization and it is worthwhile to consider a more realistic case in the next example.

Example 3: longitudinal electrostatic waves k||E in warm $T_e \neq 0$ plasma.

In this case, the **eqn. of motion** has an additional pressure term

(1) $m_e \frac{\partial}{\partial t} v_1 = q_e E_1 - \nabla p$ which needs to be transformed to yield an equation only in E and v.

For slow changes the average gas pressure is described by the adiabatic law $\bar{p} = C \cdot \bar{n}^{\gamma_{ad}}$ where the adiabatic index is found from the degrees of freedom (=f) of the gas particles as $\gamma_{ad} = \frac{f+2}{f}$ and for a simple non-relativistic gas with f=3 we get the well known result of 5/3. For a one-dimensional gas, γ_{ad}=3, and for fast (isothermal) changes the exponent will become γ=1. We will make a first order expansion for the pressure:

$$p = \bar{p} + p_1 = C \cdot (\bar{n} + n_1)^{\gamma_{ad}} = C\bar{n} \cdot \left(1 + \frac{n_1}{\bar{n}}\right)^{\gamma_{ad}} \approx C\bar{n}^{\gamma_{ad}} \cdot \left(1 + \gamma_{ad} \frac{n_1}{\bar{n}}\right)$$

or $\bar{p} + \gamma_{ad} n_1 \frac{\bar{p}}{\bar{n}} = \bar{p} + \gamma_{ad} n_1 k_B T_e$ from the use of the ideal gas law $\frac{\bar{p}}{\bar{n}} = k_B T$. We can now rewrite the pressure gradient as the gradient of the density: $\nabla p = \gamma_{ad} k_B T_e \nabla n_1$. With the help of the first order continuity eqn. $\frac{\partial n_1}{\partial t} + \bar{n}_e \nabla \cdot v_1 = 0 \quad \rightarrow \quad i\omega n_1 = \bar{n}_e \nabla \cdot v_1$ and substituting the usual $\propto$ *exp(iωt)* time dependence we obtain the linearised equation of motion:

$$i\omega m_e v_1 = q_e - \frac{i\gamma_{ad}\bar{n} \cdot k_B T_e}{\omega} \nabla(\nabla \cdot v_1)\,.$$

Let us assume that spatial variations occur only in the x-direction, then using an *exp(ikx)* ansatz will enable us to eliminate the spatial derivatives and we can finally write

$$i\omega m_e v_1 = q_e + i\gamma_{ad}\bar{n} \cdot k_B T_e v_1 \cdot \frac{k^2}{\omega}$$

With the sound speed $v_s{}^2 = \frac{\gamma_{ad}\bar{n}\, k_B T_e}{m_e}$ we get **Ohm's law** by multiplication with $n_e q_e$ and solving for v_1.

(2) $$j_x = \underbrace{-i\frac{q_e^2 \bar{n}_e}{m_e \omega^2 \left(1 - \frac{v_s{}^2 k^2}{\omega^2}\right)}}_{\sigma} \cdot E$$

again $\boldsymbol{\sigma}$ is that of a loss-less dielectric, but this time we have an additional singularity at $v_{ph}=\pm v_s$! Again using **Ampere's law**

(3) $$i\omega\varepsilon_0 E_x + j_x = 0$$

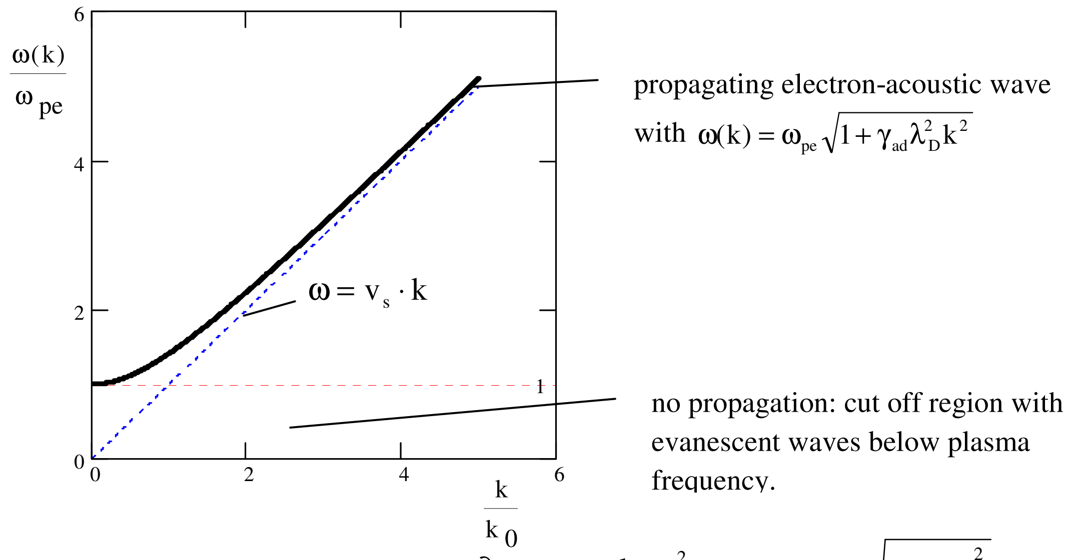
will lead us to the dispersion relation for longitudinal waves in warm plasma:

$$\omega^2 = \omega_{pe}{}^2 + \frac{\gamma_{ad}\bar{n}_e k_B T_e}{m_e} k^2 = \omega_{pe}{}^2 + v_s^2 k^2$$

It looks very similar to the dispersion relation for transverse waves, only that c has now been replaced by the sound speed v_s. The waves are therefore called **longitudinal electron-acoustic** waves (or 'Plasmons'). The Debye length λ_D is an important quantity for plasma acoustic waves. By virtue of $\lambda_D = \sqrt{\frac{\varepsilon_0 k_B T_e}{\bar{n}_e q_e^2}} = \frac{v_s}{\omega_{pe} \cdot \gamma_{ad}^{\frac{1}{2}}}$ we can write the dispersion relation now simply as

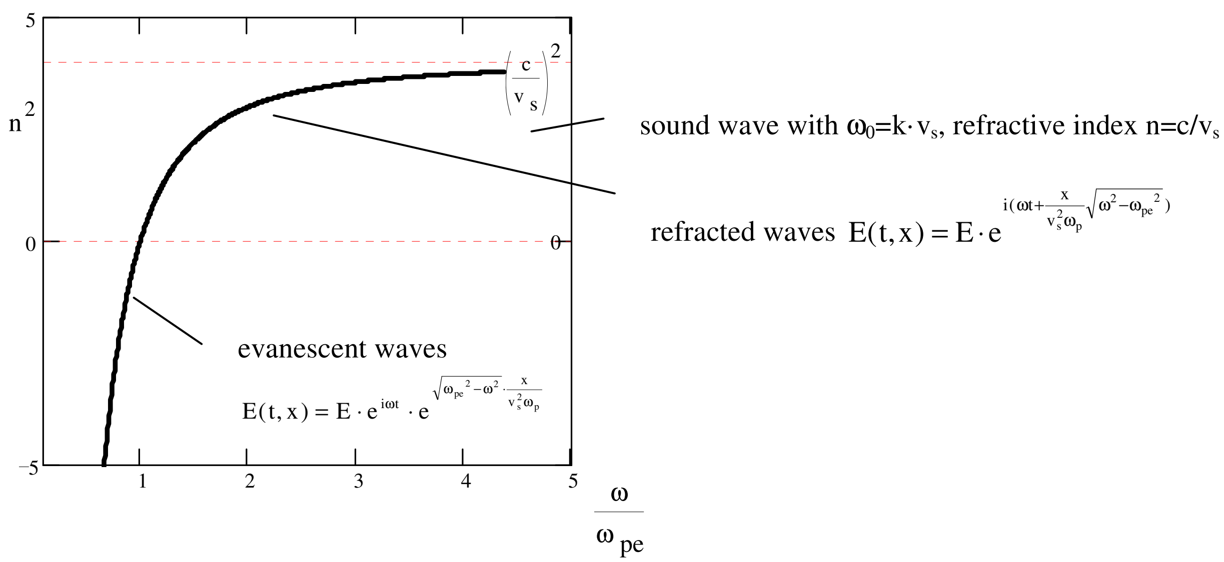
$$\omega^2 = \omega_{pe}{}^2\left(1 + \gamma_{ad}\lambda_D^2 k^2\right)$$

and we note that the Debye length determines how the plasma will react to short waves.



One notes that the group velocity $v_{gr} = \dfrac{\partial\omega}{\partial k} = \dfrac{k\ v_s^{\ 2}}{\sqrt{k^2 v_s^{\ 2} + \omega_{pe}^{\ 2}}} \ = v_s \cdot \sqrt{1 - \dfrac{\omega_{pe}^{\ 2}}{\omega^{\ 2}}}$ will approach the sound speed v_s for high frequencies and so will the phase velocity $v_{ph} = \dfrac{\omega}{k} = \sqrt{v_s^{\ 2} + \dfrac{\omega_{pe}^{\ 2}}{k^2}}$. We find a simple expression for the squared refractive index:

$$n^2 = \frac{c^2}{v_s^2}\left(1 - \frac{\omega_p^{\ 2}}{\omega^{\ 2}}\right).$$

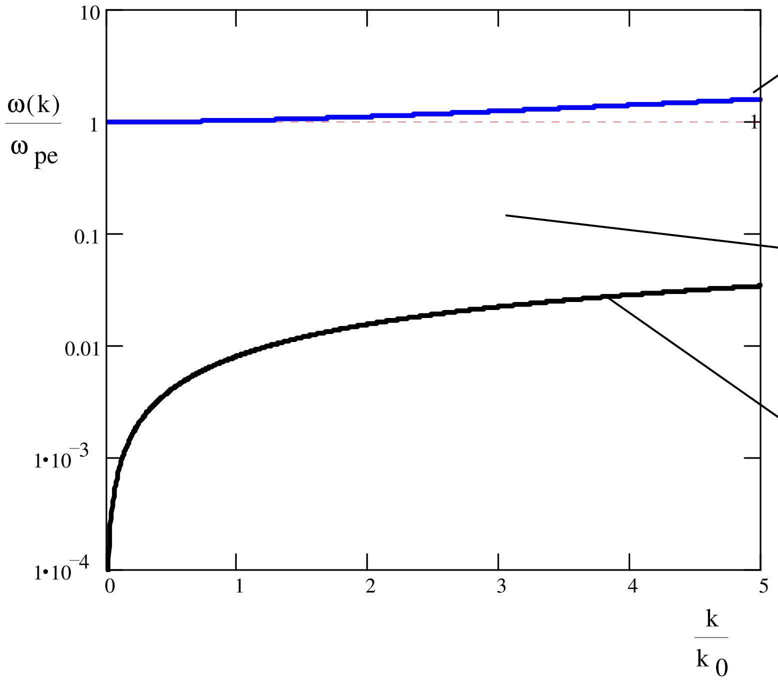


Example 4: warm ion + electron plasma

We can take the additional motion of ions into account by introducing their sound speed $v_{si}{}^2 = \dfrac{\gamma_{ad}\bar{n}\,k_B T_i}{m_i}$ and their plasma frequency $\omega_{pi} = \sqrt{\dfrac{q_e{}^2 n_i}{\varepsilon_0 m_i}}$ and use the same procedure as in the previous examples to derive a dispersion relation which in this case reads:

$$1 = \frac{\omega_{pe}^2}{\omega^2 - v_{se}^2 k^2} + \frac{\omega_{pi}^2}{\omega^2 - v_{si}^2 k^2}$$

It has two branches



propagating **electron-acoustic wave** with

$\omega_{eac}(k) = (\omega_{pe}{}^2 + \omega_{pi}{}^2)\sqrt{1+\gamma_{ad}\lambda_D^2 k^2}$

forbidden region:

No wave propagation for $\omega_{iac} < \omega < \omega_{eac}$

propagating **ion-acoustic** wave with $\omega_{iac}(k) \cong v_{si} \cdot k$.

corresponding to electron and ion pressure waves (**electron and ion acoustic waves**).

Waves in Magnetized Plasma

Longitudinal waves

a) **for k||B$_0$ and E||k** the v x B term in the equation of motion vanishes and we have the same cases as before without the magnetic field component. Plasma oscillations can happen and Langmuir waves (see example 1) can propagate parallel to the magnetic field without being affected by its presence.

b) **for E⊥B$_0$ as i.e. B$_0$=(0,0,B)** the equation of motion $i\omega m_e \cdot v_1 = q_e E_1 + v_1 \times B_0$ leads to a tensor for the conductivity:

$$\sigma_{i,j} = -i\frac{q_e^2 n_e}{\omega \cdot m_e}\frac{1}{1-\frac{\omega_g^2}{\omega^2}}\begin{pmatrix} 1 & -i\frac{\omega_g}{\omega} & 0 \\ i\frac{\omega_g}{\omega} & 1 & 0 \\ 0 & 0 & 1-\frac{\omega_{pe}^2}{\omega^2} \end{pmatrix}$$

and as a consequence the dielectric properties of a magnetized plasma are also anisotropic as expressed by the dielectric tensor:

$$\varepsilon_{i,j} = \begin{pmatrix} \varepsilon_1 & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_0 \end{pmatrix} = \begin{pmatrix} 1-\frac{\omega_{pe}^2}{\omega^2-\omega_g^2} & -i\frac{\omega_g}{\omega}\cdot\frac{\omega_{pe}^2}{\omega^2-\omega_g^2} & 0 \\ i\frac{\omega_g}{\omega}\cdot\frac{\omega_{pe}^2}{\omega^2-\omega_g^2} & 1-\frac{\omega_{pe}^2}{\omega^2-\omega_g^2} & 0 \\ 0 & 0 & 1-\frac{\omega_{pe}^2}{\omega^2} \end{pmatrix}$$

We note that there will be resonances at the plasma frequency and at the gyro frequency. In order to get a general expression for a dispersion relation we use the linearised Maxwell equations $\nabla \times E = -i\omega B$ and $\nabla \times \vec{B} = i\omega\mu_0\varepsilon_0\bar{\varepsilon}_r \cdot \vec{E}$ which we can combine together with the spatial depency $\propto e^{ik\cdot x}$ to yield the dispersion relation in vector notation

$$\frac{-c^2k^2}{\omega^2}(\vec{k}_e \times \vec{k}_e \times \vec{E}) = \bar{\varepsilon}_r \cdot \vec{E}$$

with $\vec{k}_e = \frac{\vec{k}}{|\vec{k}|}$ the unit vector in the propagation direction. It is more convenient to write this in component notation:

$$\left(\underbrace{\varepsilon_{i,j} - \frac{c^2k^2}{\omega^2}\left(\delta_{i,j} - k_{e_j}k_{e_i}\right)}_{D}\right)\cdot E_j = 0$$

For this equation to hold for arbitrary E, the determinant of the bracket will have to vanish.

Hence the general dispersio relation for the magnetized plasma is given by

det(D)=0

Waves are linear eigenmodes of D at ω(k) and wave vector k. The solutions are frequently complicated and depend on frequency and propagation directions.

For a cold plasma with heavy ions and wave propagation at an angle θ to B given by $\vec{k}_e = (0, \sin\theta, \cos\theta)$, we obtain a dispersion relation of the form

$$\begin{vmatrix} \varepsilon_1 - \dfrac{c^2k^2}{\omega^2} & -i\varepsilon_2 & 0 \\ i\varepsilon_2 & \varepsilon_1 - \dfrac{c^2k^2}{\omega^2}\cos^2\theta & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ 0 & \dfrac{c^2k^2}{\omega^2}\sin\theta\cos\theta & \varepsilon_0 - \dfrac{c^2k^2}{\omega^2}\sin^2\theta \end{vmatrix} = 0$$

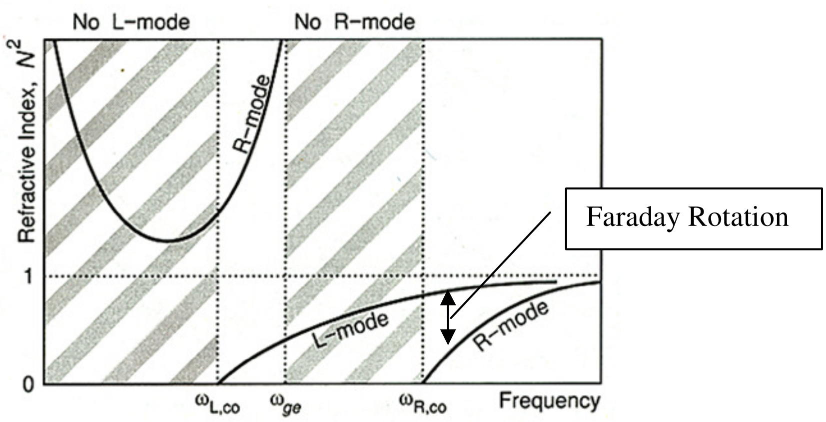
where we have used ε_0, ε_1 and ε_2 as abbreviations given in the definition of the dielectric tensor above. **Det(D)=0** written as a polynomial equation in ω,k and θ is also known as the Hartree-Appleton Formula for wave propagation in magnetized plasma.

For parallel propagation (k||B → θ=0) we obtain two solutions:

$$n_{R,L}{}^2 = \frac{c^2k^2}{\omega^2} = \varepsilon_1 \pm \varepsilon_2 = 1 - \frac{\omega_{pe}{}^2}{\omega(\omega \pm \omega_g)}$$

The **corresponding waves are circular polarised** and the + sign stands for the left hand circular (LHC) polarisation and the – sign for the right hand circular polarisation (RHC). The graph from Lesch below

Refractive index for parallel R- and L-waves

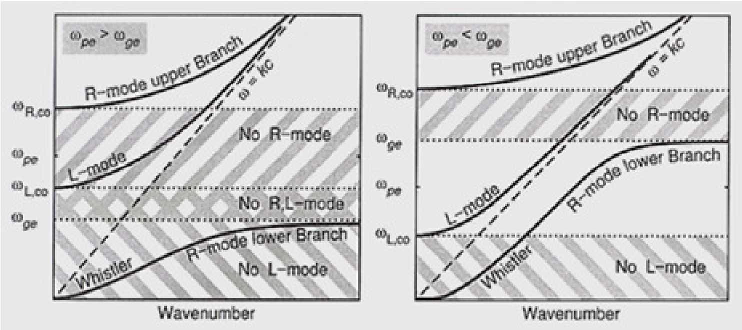


There is no wave propagation for $N^2 < 0$, regions which are called ***stop bands*** or domains where the waves are ***evanescent***.

shows that R-modes can propagate up to the resonance at ω_g and again above a forbidden region for RHC waves from ω_g up to a cut-off frequency of $\omega_{R,co} = \dfrac{\omega_g}{2} + \sqrt{\omega_{pe}{}^2 + \dfrac{\omega_g{}^2}{4}}$. The

LHC however, has only one propagating branch above $\omega_{L,co} = \frac{-\omega_g}{2} + \sqrt{\omega_{pe}{}^2 + \frac{\omega_g{}^2}{4}}$. LHC and RHC waves can propagate at frequencies above $\omega_{R,co}$, but do so with different velocities leading to a rotation of the polarisation plane (Faraday rotation).

Dispersion branches for parallel R- and L-waves



The dispersion branches are for a ***dense*** (left) and ***dilute*** (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than *c*. Note also that the R- and L-waves can not penetrate below their cut-off frequencies. The R-mode branches are separated by ***stop bands.***

The two diagrams above (by Lesch) show the behaviour of $\omega(k)$ for dense ($\omega_{pe}>\omega_g$) and dilute ($\omega_{pe}<\omega_g$) plasma.

Whistler waves with $\omega<<\omega_g$ are highly dispersive waves with $\omega(k) \approx \frac{k^2c^2}{\omega_{pe}{}^2}\omega_g$. Their group velocity $v_g \propto \frac{1}{\lambda}$. They are exited by lightening in planetary magnetospheres (Earth, Jupiter) and characterised by a high → low whistling sound when received.

The difference in refractive index for RHC and LHC waves above $\omega_{R,co}$ is approximated by

$$n_R - n_L \cong \frac{\omega_{pe}{}^2\omega_g}{\omega^3} = \frac{q_e{}^3}{\varepsilon_0 m_e{}^2}\frac{n_e B}{\omega^3}$$

and that leads directly to the difference in phase angle for Faraday Rotation:

$$\chi = \frac{\omega}{c}\Delta n = \frac{q_e{}^3}{2\cdot\varepsilon_0 m_e{}^2 c}\frac{1}{\omega^2}\int_0^d n_e(x)B(x)dx = \quad 2.62\cdot 10^{13}\lambda^{-2}\int_0^d n_e(x)B(x)dx$$

transverse waves

c) **Propagation k⊥B with E||B:** transverse waves (independent of B) are called ordinary modes (**o-modes**) or **Langmuir modes** and their dispersion is given by the Langmuir solution (see **example 1**):

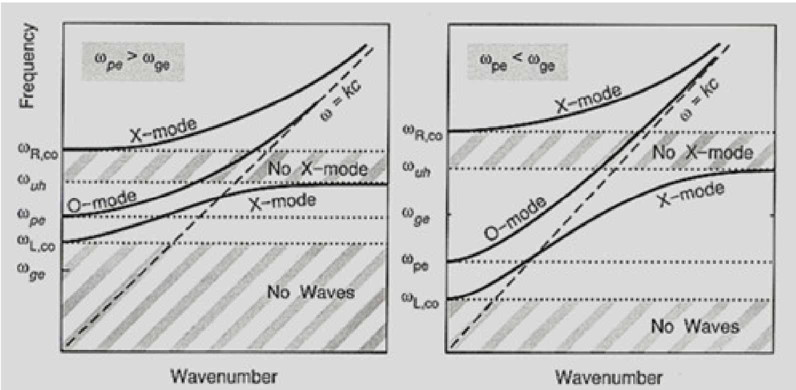
$$\omega_O(k) = \sqrt{c^2k^2 + \omega_{pe}^2}$$

d) **Propagation k⊥B with E⊥B:** a mixture of transverse and longitudinal waves is called the extra-ordinary mode (**x-mode**). Its dispersion relation is:

$$k_\perp^{\ 2}c^2 = \frac{(\omega^2 - \omega_{R,co}^{\ \ 2})(\omega^2 - \omega_{L,co}^{\ \ 2})}{\omega^2 - \omega_{uh}^{\ \ 2}}$$

and shown in the following graph from Lesch:

Dispersion for perpendicular O- and X-waves



The dispersion branches are for a ***dense*** (left) and ***dilute*** (right) plasma. Note the tangents to all curves, indicating that the group velocity is always smaller than *c*. Note that the O- and X-waves can not penetrate below the cut-off frequencies. The X-mode branches are separated by ***stop bands.***

The upper hybrid frequency is a combination of plasma and gyro frequency $\omega_{uh} = \sqrt{\omega_{pe}^{\ \ 2} + \omega_{ge}^{\ \ 2}}$ and the two cut-off frequencies are those given in **case (b)** above. There is no wave propagation below the lower cut-off frequency $\omega_{L,co}$, which is usually comparable to the plasma frequency ω_{pe}. The different propagation properties of o and x-modes are important in the strongly magnetized and dilute plasma of the inner pulsar magnetosphere and are thought to give rise to some of the pulse profile and polarization features of pulsar radio emission.

e) **Alfven waves**: Adding the effects of the motion of ions to the equation adds complexity to the dispersion relations but for θ=0 we get circular polarized waves with:

$$n_{R,L}^{\ \ 2} = 1 - \frac{\omega_{pe}^{\ \ 2}}{\omega(\omega \mp \omega_{ge})} - \frac{\omega_{pi}^{\ \ 2}}{\omega(\omega \pm \omega_{gi})}$$

the plasma is taken to be neutral and hence $\omega_{pe}^{\ \ 2}\omega_{gi} - \omega_{pi}^{\ \ 2}\omega_{ge} = 0$ which enables us to simplify

$$n_{R,L}{}^2 = 1 - \frac{\omega_{pe}{}^2 + \omega_{pi}{}^2}{(\omega \mp \omega_{ge})(\omega \pm \omega_{gi})}$$

for very low frequencies $\omega<<\omega_{pi}$ (the magneto-hydrodynamic regime), both branches of the dispersion relation reduce to

$$n^2 = 1 + \frac{\omega_{pe}{}^2}{\omega_{ge}\omega_{gi}} = 1 + \frac{m_i n_e}{\varepsilon_0 B^2}$$

which is independent of ω and k , and corresponds to waves with a constant phase velocity

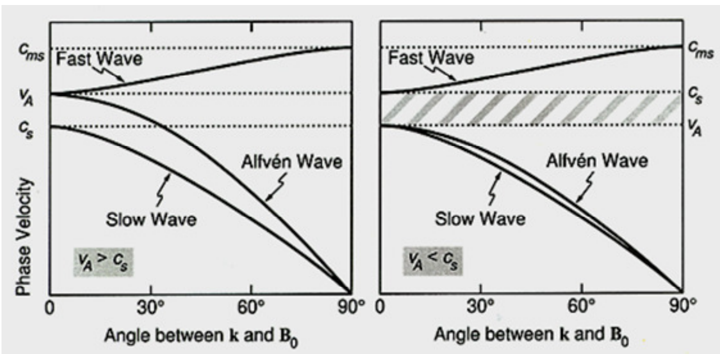
$$\frac{c}{n} = \frac{c}{\sqrt{1 + \frac{m_i n_e}{\varepsilon_0 B^2}}} = \frac{v_A}{\sqrt{1 + \frac{v_A{}^2}{c^2}}}$$

with $v_A = \frac{B}{\sqrt{\mu_0 m_i n_e}}$ being the Alfven velocity. Both (LHC and RHC) waves propagate with the same velocity along B giving rise to a linear polarised wave. The particles show periodic ExB drift oscillations, the waves are therefore called torsional (Alfven) waves. We also expect magnetic pressure waves to occur as the total pressure in a magnetized plasma is given by $p + \frac{B^2}{2\mu_0}$. The procedure outlined in **example 3** can be used to derive dispersion relations for the fast and slow magneto-sonic waves:

$$\omega_{ms}{}^2 = \frac{k^2}{2}(c_{ms}{}^2 \pm \sqrt{(v_A{}^2 - c_s{}^2)^2 + 4 v_A{}^2 c_s{}^2 \sin^2\theta})$$

with $c_{ms}{}^2 = c_s{}^2 + v_A{}^2$.

Dependence of phase velocity on propagation angle

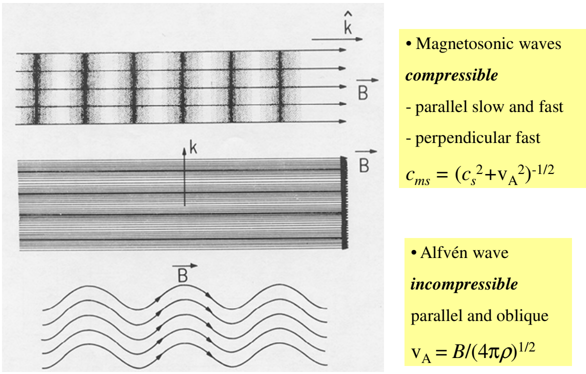


(Lesch)

For $\theta=90°$ we have only one (fast) mode $\omega=kc_{ms}$, but for $\theta=90°$ we get two solutions

$$\omega_{ms}{}^2 = k^2(c_s{}^2 + v_A{}^2 \pm (v_A{}^2 - c_s{}^2)) \quad = \begin{cases} k^2 c_s{}^2 & \text{sound wave} \\ k^2 v_A{}^2 & \text{Alfven wave} \end{cases}$$

Magnetohydrodynamic waves



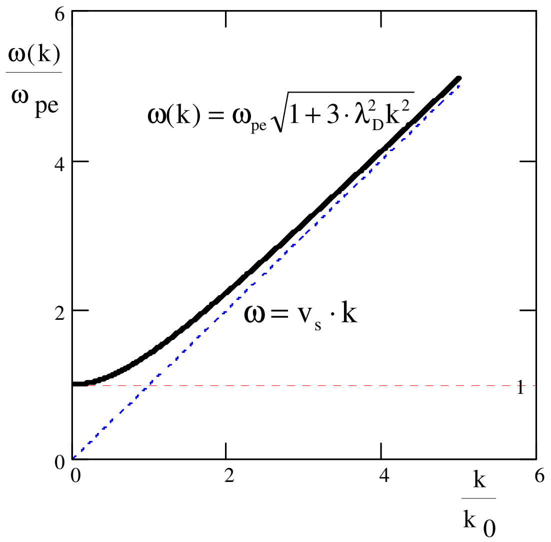
(Lesch)

Here the Alfven wave is an incompressible but distortional wave flexing the magnetic field lines like a vibrating string. In a vibrating string, the sound speed of the waves is given by the root of the ratio of tension T to mass density ρ: $v_s = \sqrt{\frac{T}{\rho}}$. In an analogy one identifies the tension $T = \frac{B^2}{\mu_0}$ in $v_A = \sqrt{\frac{B^2}{\mu_0} \cdot \frac{1}{m_i n_e}}$, the mass density is given as usual as $\rho=m_i n_e$.

Kinetic Theory

Motivation:

Consider the simple case of **longitudinal electron-acoustic** waves with one degree of freedom:

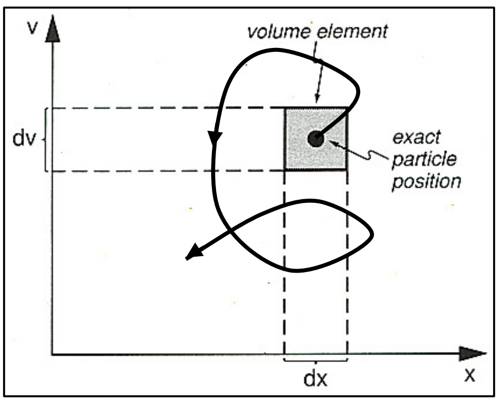


The dispersion relation was derived using linear theory and the assumption of idealised ordered particle motions. Evidently there is no dissipation and such waves could exist forever. However the second law of thermodynamics implied that highly ordered motions will not prevail indefinitely and we will have to consider several basic questions:

How long will they live and how far can they propagate?

In short, we have to analyse the stability of plasma oscillations and waves. An instability can lead to growth or damping of waves. The growth rate of waves is conveniently expressed as an imaginary part of the frequency: $A(t) = A_0 e^{i(\omega + i\gamma)t}$. The imaginary part γ of the complex frequency $\varpi = \omega + i\gamma$ will describe growing or damped waves, depending on its sign.

We need to account for the statistical properties of the particles in more sophisticated ways than the use of coarse-grained ensemble averages like T or λ_D. Looking at an individual particle, we find, that one describes it by its three spatial (**x**) and three velocity coordinates (**v**), together spanning a 6-dimensional *phase space*. Each particle will move on a trajectory in phase space as a result of its interaction with its neighbours and ambient force fields. For an individual particle we can write $f_i(\vec{x}, \vec{v}, t) = \delta(\vec{v} - \vec{v}_i(t)) \cdot \delta(\vec{x} - \vec{x}_i(t))$ as the phase-space density. Its sum over all N particles $f(\vec{x}, \vec{v}, t) = \sum_{i=1}^{N} f_i(\vec{x}, \vec{v}, t)$ is called the particle density distribution (in phase space) and it contains all the information about the plasma. However with $N > 10^{30}$ for ordinary plasma this becomes too unwieldy. However the modern particle-in-cell simulations try to model a plasma by employing a coarser grid for the phase-space and by combining many particles (at least as many as N_D) into clusters called 'superparticles'. The fields generated by the superparticles and their individual motion on the grid can approximate the non-linear plasma



(Lesch)

behaviour. The reduction in dimension makes some problems tractable, but still computationally very expensive.

Kinetic plasma theory approaches the problem by describing properties and evolution of the particle distribution function (pdf) and by using averages and moments that connect to observables.

Integration of the pdf over velocity space yields the particle density as a function of location and time:

$$n(\vec{x}, t) = \int f \cdot d^3 v$$

The average velocity is given as the first moment of the velocity distribution:

$$< \vec{v}(\vec{x}, t) > = \frac{\int \vec{v} \cdot f \; d^3 v}{n(\vec{x}, t)}$$

and the average particle energy and temperature is found from the second moment:

$$\frac{3}{2} n k_B T = \int \frac{m}{2} v^2 \cdot f \; d^3 v$$

How does f(x,v,t) look like and how does it evolve with time? We make a couple of simplifying assumptions:

1. The total number of particles is conserved: there is no pair creation, hence T< 500 keV. There will be no recombination of ions: hence T>10-20 eV. We are not investigating cold or extremely relativistic plasma.
2. Particles do not instantaneously change position of velocity: Quantum mechanical effects have to be negligible (only non-degenerate plasma).

As a result, we know that there will be smooth flow of 'phase fluid' describable by a continuity (Liouville) equation:

$$\frac{\partial f}{\partial t} + \nabla_6 (u \cdot f) = 0$$

with ∇_6 being the derivative in all six phase-space directions and u the 6-D vector of time derivatives $(\frac{\partial \vec{x}}{\partial t}, \frac{\partial \vec{v}}{\partial t})$. Written in component notation, the Liouville equation reads

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_\mu}\left(f \frac{\partial x_\mu}{\partial t} \right) + \frac{\partial}{\partial v_\mu}\left(f \frac{\partial v_\mu}{\partial t} \right) = 0$$

With $\frac{\partial\vec{x}}{\partial t}=\vec{v}$, homogeneity $\frac{\partial\vec{v}}{\partial\vec{x}}=0$ and by using the expression for the Lorentz force $\frac{\partial\vec{v}}{\partial t}=\frac{q_e}{m}(\vec{E}+\vec{v}\times\vec{B})$ we get the **Vlasov equation**:

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial\vec{x}}+\frac{q_e}{m}(\vec{E}+\vec{v}\times\vec{B})\cdot\frac{\partial f}{\partial\vec{v}}=0$$

It is a first order linear differential equation describing the evolution of the particle distribution function in given electro-magnetic fields and in the absence of collisions. Collision effects can be included in the analysis by replacing $0\rightarrow\left.\frac{\partial f}{\partial t}\right|_{coll}$. Various approximations to the collision term have been made in the past. Constructing the additional collision term from the probabilities of two particle collisions gives the Boltzmann equation. In the Lenard-Balescu equation, only Coulomb collisions within the Debye-sphere of radius λ_D are considered and the Fokker-Plank approximation uses an average over the first order variations of the collective forces: $\left.\frac{\partial f}{\partial t}\right|_{coll}=\frac{q_e}{m}<(\delta\vec{E}+\vec{v}\times\delta\vec{B})\cdot\frac{\partial f_1}{\partial\vec{v}}>$.

We will however consider only the collision less case for our subsequent analysis. The Vlasov equation may be linear, but the e.m. self-interaction described by the fields through Maxwell's equations will make it a non-linear system of equations. In addition we have

1.) Poisson's equation: $\nabla\vec{E}=\frac{q_e n}{\varepsilon_0}=\sum_{species}q_{species}\int f\cdot d^3v$

2.) Ampere's equation $\nabla\times\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}+\underbrace{\mu_0\sum_{species}q_{species}\int f\cdot d^3v}_{=i}$

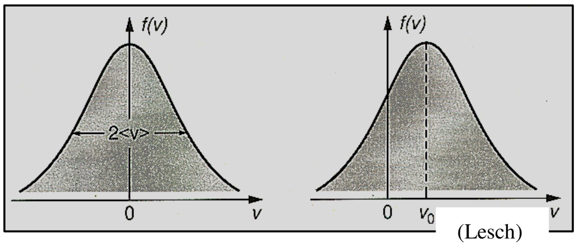
which describe how the fields are generated by the particles described through their pdf.

(The other two Maxwell equations, the induction eqn. and the source-free magnetic field condition are pure field equations which do not directly involve any charges.).

In kinetic theory, the plasma is described by a non-linear set of integro-differential equations in six dimension plus time. For their solution one requires boundary conditions on E and B as external sources.

The Maxwellian distribution is an equilibrium solution of the Vlasov equation in the absence of external fields E and B:

$$f(v) = n\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} e^{-\frac{m(v-v_0)^2}{2k_B T}}$$



(Lesch)

The average velocity spread (variance) is,

$\langle v\rangle = (2k_BT/m)^{1/2}$, and the mean drift velocity, $\mathbf{v}_0$.

Note: Any function f(W) of a constant of motion or invariant W in phase-space will satisfy the Vlasov equation. Because of that, not only a Gaussian, but also any power-law or other function of particle energy will be a possible solution of the Vlasov equation in the absence of external electromagnetic fields (Jeans Theorem).

Example: Shielding of a point charge

In the presence of an electric field $E = -\nabla\Phi(x)$ given through the electric potential $\Phi(x)$, the total energy $w = \frac{1}{2}mv^2 + q_e\Phi$ is a constant of motion. Hence $f \propto e^{\frac{-\frac{1}{2}mv^2 - q_e\Phi}{kT}}$ will be solution of the Vlasov equation. The density is then given as $n(\vec{x}) = \int f \cdot d^3v = n_0 e^{\frac{-q_e\Phi}{kT}}$, in its form similar to the familiar barometric equation. Let us consider a spherical symmetric case, but with two species i,e of oppositely charged particles. Their densities are: $n_i(\vec{x}) = n_0 e^{\frac{-q_e\Phi}{kT}}$ and $n_e(\vec{x}) = n_0 e^{\frac{q_e\Phi}{kT}}$ and the Poisson equation can be written as $\nabla^2\Phi = -\frac{q_e}{\varepsilon_0}(n_e - n_i) = \frac{q_e n_0}{\varepsilon_0}\sinh\underbrace{\left(\frac{q_e\Phi}{kT}\right)}_{\Psi}$. Using the new variable Ψ results in $\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Psi = \frac{q_e^2 n_0}{\varepsilon_0 kT}\sinh(\Psi) \cong \frac{2}{\lambda_D^2}\Psi$ for small Ψ. The solution for $\Phi \to 0$ when $r \to \infty$ and $\Phi(r) = \frac{q_e}{4\pi\varepsilon_0 r}$ for $(r << \lambda_D)$ is the familiar Debye-Hückel Potential $\Phi(r) = \frac{q_e}{4\pi\varepsilon_0 r}e^{-\frac{\sqrt{2}r}{\lambda_D}}$.

Application to plasma oscillations

Take a uniform, steady-state plasma described by $f_0(v)$ (i.e. a Maxwellian) and allow small perturbations. We assume that there are no background fields. The simplest kind of waves are then electrostatic plasma oscillations of one species. We are making the usual first order approximation $f = f_0 + f_1 e^{-i(\omega t - kx)}$ for all quantities (E, n, j).

The Vlasov equation $\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{q_e}{m}(\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f}{\partial \vec{v}} = 0$ with this ansatz and neglecting v×B and $\frac{\partial f_1}{\partial \vec{v}}$ is then simply $-i\omega f_1 + i\vec{k} \cdot \vec{v}\; f_1 = -\frac{q_e}{m}\vec{E}_1 \frac{\partial f_0}{\partial \vec{v}}$ which we can solve for f_1: $f_1 = -\frac{i\frac{q_e}{m}E_1\frac{\partial f_0}{\partial \vec{v}}}{\omega - \vec{k} \cdot \vec{v}}$.

Hence the charge density is $n_1 = -i\frac{q_e}{m}\int_V \frac{E_1 \frac{\partial f_0}{\partial \vec{v}}}{\omega - \vec{k} \cdot \vec{v}} d^3v$

Now assume that E||v and use the linearised Poisson equation $\vec{k} \cdot \vec{E} = -i\frac{q_e n_1}{\varepsilon_0}$ to get

$$i\varepsilon_0 \vec{k} \cdot \vec{E}_1 = \sum_{species} -i\frac{q_{(s)}^{\;2}}{m_{(s)}}\vec{E}_1 \int_V \frac{\frac{\partial f_0}{\partial \vec{v}_{\parallel}}}{\omega - \vec{k} \cdot \vec{v}} d^3v$$

which for non-trivial E requires

$$\sum_{species} -i\frac{q_{(s)}^{\;2}}{m_{(s)}k^2}\vec{k} \int_V \frac{\frac{\partial f_0}{\partial \vec{v}_{\parallel}}}{\omega - \vec{k} \cdot \vec{v}} d^3v = -1$$

This is dispersion relation and has a singularity at ω=k·v. It allows us to find ω(k) for a given the unperturbed particle distribution function f_0 .By partial integration we get

$$\sum_{species} -i\frac{q_{(s)}^{\;2}}{m_{(s)}k^2}\vec{k} \int_V \frac{f_0}{(\omega - \vec{k} \cdot \vec{v})^2} d^3v = +1$$

which for a cold plasma (v=0) is $\sum_{species} \frac{q_{(s)}^{\;2} n_{(s)}}{\varepsilon_0 m_{(s)} \omega} = 1$ or just the dispersion relation $\omega^2 = \sum_{species} \omega_{p(s)}^{\;2}$ for plasma oscillations at the hybrid frequency $\varpi = \sqrt{\sum_{species} \omega_{p(s)}^{\;2}}$.

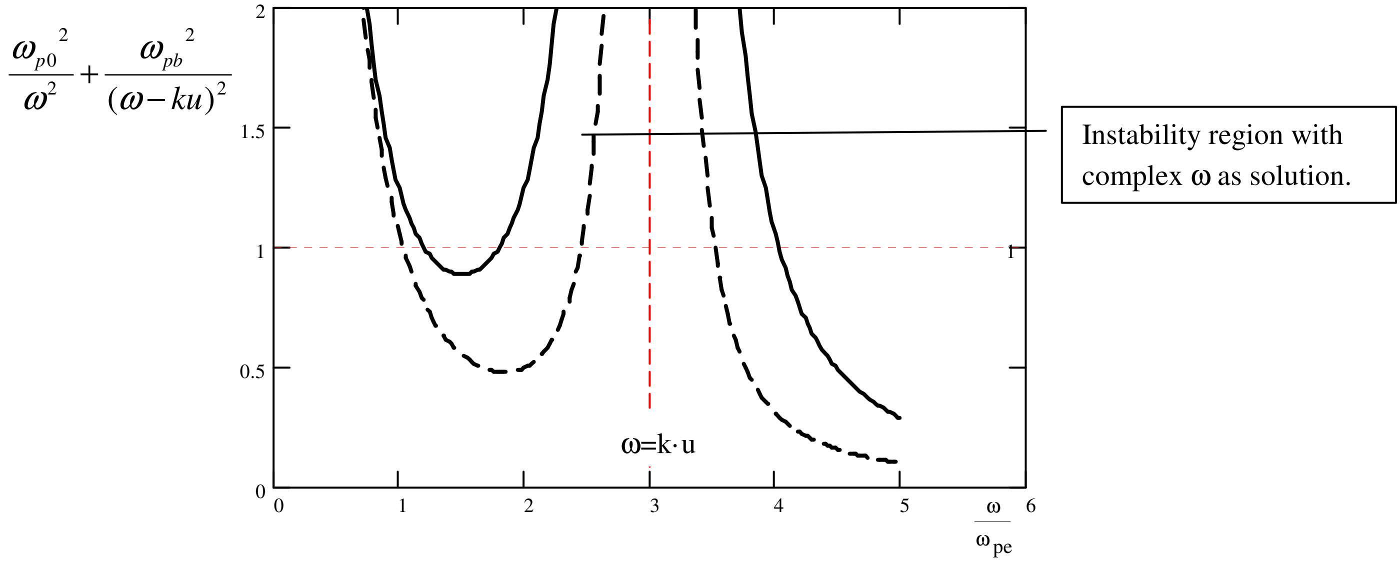
For a one component (electron) cold plasma, where one part is at rest(v=0), and the other (the beam) moving at v=u we get from above

$$\frac{q_e^{\;2} n_0}{\varepsilon_0 m_e}\frac{1}{\omega^2} + \frac{q_e^{\;2} n_b}{\varepsilon_0 m_e}\frac{1}{(\omega - ku)^2} = 1$$

which can be written as

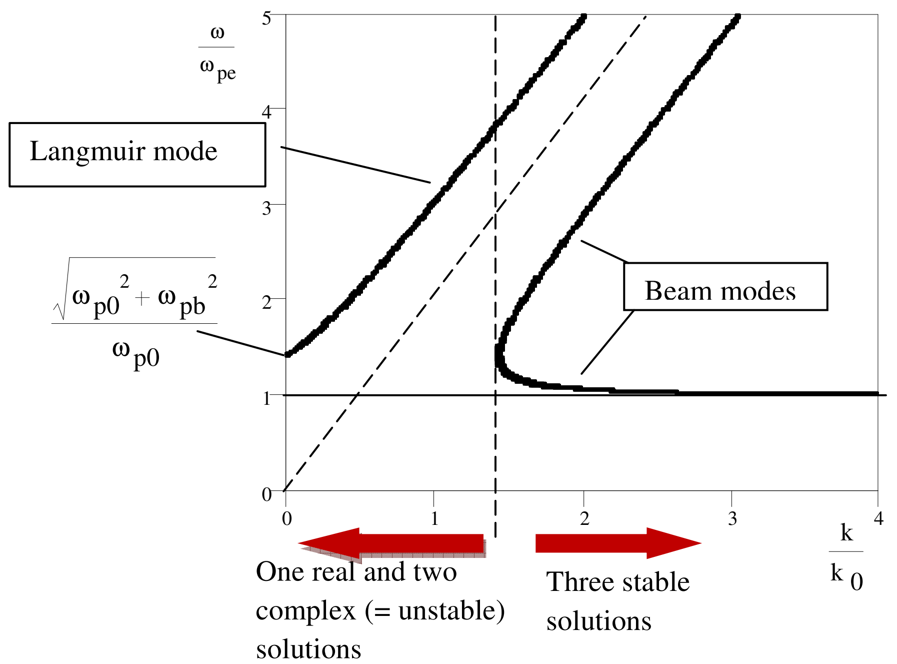
$$\frac{\omega_{p0}^{\;2}}{\omega^2} + \frac{\omega_{pb}^{\;2}}{(\omega - ku)^2} = 1$$

It has two resonances, one near the plasma frequency of the background (Langmuir part) and another describing the additional beam modes. The graph shows the behaviour of the left hand side of the equation for $n_b=n_0$ (solid) and for $n_b=0.5n_0$ (dashed) with a resonance at $k\cdot u=3\omega_0$.



There are three real positive solutions for w, for any $n_b \le n_0$. The first two separate more with decreasing beam density n_b, but the beam mode solutions (second and third) draw closer to the resonance $\omega=ku$ for smaller n_b. Allowing for complex $\omega(k)$ will provide solutions in some regions where the real part of $\frac{\omega_{p0}^2}{\omega^2}+\frac{\omega_{pb}^2}{(\omega-ku)^2}>1$, leading unstable solutions (**<u>streaming instability</u>**).

The three positive branches of the dispersion relation for the $\mathbf{n_b=n_0}$ case are show in the plot below:



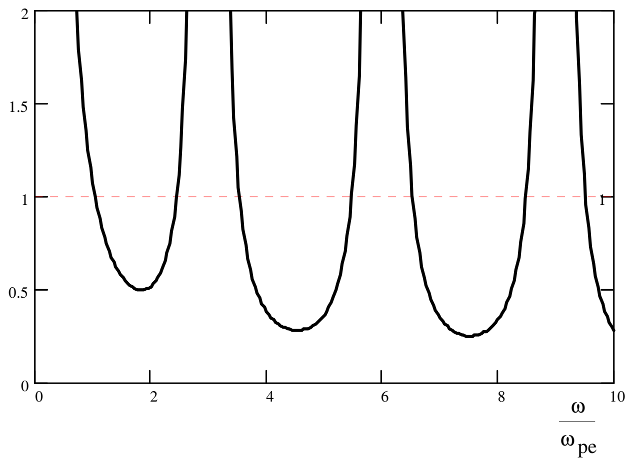
The dispersion relation is a forth-order equation having four roots, one of them with negative ω. For the simple case $\mathbf{n_b=n_0}$, they are given by

$$\omega = \frac{k\,u}{2} \pm \frac{1}{2}\left(\underbrace{4\,\omega_p^{\;2} + k^2u^2 \pm 4\left(\omega_p^{\;4} + \omega_p^{\;2}k^2u^2\right)^{\frac{1}{2}}}_{X_\pm} \right)^{\frac{1}{2}}$$

The radicand with the negative sign ($\mathbf{X_-}$) will be less than 0 when $k<\omega_p/u$, leading to complex valued frequencies $\omega(k) = \frac{k\,u}{2} \pm \frac{i}{2}\sqrt{-X_-}$. The imaginary part of the frequency describes the growth or damping rate of a wave given by $e^{-\frac{t}{\tau}} \cdot e^{-i\omega t}$ with $\tau^{-1} = \gamma(k) = \mathrm{Im}(\omega(k))$.

The growth or damping rate is $\gamma(k) = \pm\left(\left(\omega_p^{\;4} + \omega_p^{\;2}k^2u^2\right)^{\frac{1}{2}} - \omega_p^{\;2} - \frac{k^2u^2}{4} \right)^{\frac{1}{2}}$ in our discussed example of the **two-stream instability** with one stream at rest. The fastest growth happens at $k = \sqrt{3}\frac{\omega_p}{u}$ where $\gamma(\sqrt{3}\frac{\omega_p}{u}) = \frac{\omega_p}{2}$, leading to a characteristic time of two periods for growth or decay. In a more complex scenario, one may have to deal with many streams at different velocities. The dispersion relation will have additional terms and more instability regions.

$$1 = \sum_j \frac{\omega_{pj}^{\;2}}{(\omega - k\cdot u_j)^2}$$ ➔



The two-stream instability is an important example for plasma wave excitation in cases of non-equilibrium momentum velocity distributions of particles. Its growth is rapid so that the linear approximation constraints used in the derivation are violated after a few plasma periods. One must note that the analysis shown can only indicate where such instabilities are expected and how fast small perturbations are expected to grow in such cases.

Landau damping for a 1-dim. warm single component plasma.

Solving the dispersion relation $\frac{\omega_{pe}^{2}}{n_0 k^2}\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - k\cdot v}dv = 1$ is difficult for general $f_0(v)$ because of the (simple) singularity at $\omega = k\cdot v$. Following Landau (Landau&Lifshitz Vol. V, 1976) we integrate in the complex plane around the singularity and apply the residue theorem:

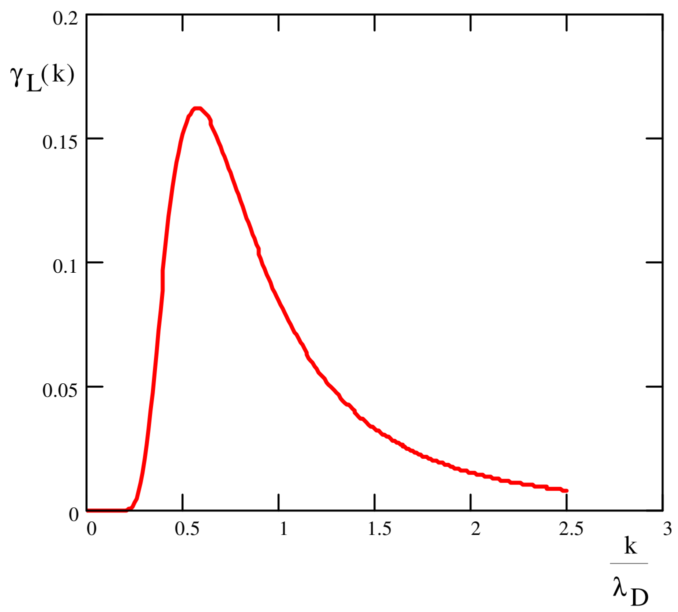
$$1 = \frac{\omega_{pe}^{2}}{n_0 k^2}\left(\lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\frac{\omega}{k}-\varepsilon}\frac{\frac{\partial f_0}{\partial v}}{\omega - k\cdot v}dv + \int_{\frac{\omega}{k}+\varepsilon}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - k\cdot v}dv\right) + i\pi\left.\frac{\partial f_0}{\partial v}\right|_{v=\frac{\omega}{k}}\right)$$

The limit of the sum of the two separate integrals that exclude the singularity is called the Cauchy principal value $\Pr\int_{-\infty}^{\infty}\frac{\frac{\partial f_0}{\partial v}}{\omega - k\cdot v}dv$ of the integral and $i\pi\left.\frac{\partial f_0}{\partial v}\right|_{v=\frac{\omega}{k}}$ its residuum. Expanding the principal value in the dispersion relation for a one dimensional Maxwellian distribution we get a complex solution for the frequency:

$$\omega(k) = \omega_{pe}\left(1+\frac{3}{2}k^2\lambda_D^{2}\right)^{\frac{1}{2}} + i\pi\frac{\omega_{pe}^{2}}{k^2}\left.\frac{\partial f_0}{\partial v}\right|_{v=\frac{\omega}{k}}$$

where the imaginary part yields the **Landau damping rate** $\gamma_L(k) = \sqrt{\frac{\pi}{8}}\frac{\omega_{pe}^{3}}{k^3\lambda_D^{3}}e^{-\frac{1}{2k^2\lambda_D^2}-\frac{3}{2}}$

$\gamma_L(k)$ is shown in the next graph:

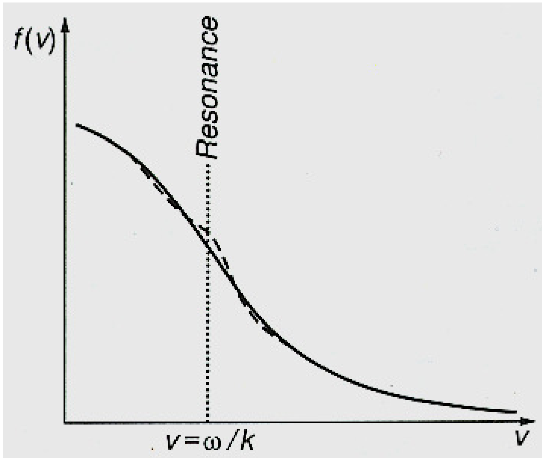


$\gamma_L(k)$ has a maximum of about 0.16 at $k=0.577\cdot\lambda_D$ which means that typical decay times are about 6-8 plasma periods. **Landau damping** causes Langmuir waves in a thermal plasma to be damped near the resonance for $T_e>0$ and $\frac{\partial f_0}{\partial v} < 0$. The same mechanism will however result in wave amplification and oscillations at any $\omega=k\cdot v$ for distributions where $\frac{\partial f_0}{\partial v} > 0$.

The slope of the distribution function is decisive for the growth or decay of waves at the Landau resonance $\omega=k\cdot v$. Positive slopes imply amplification, negative slopes result in damping of waves.

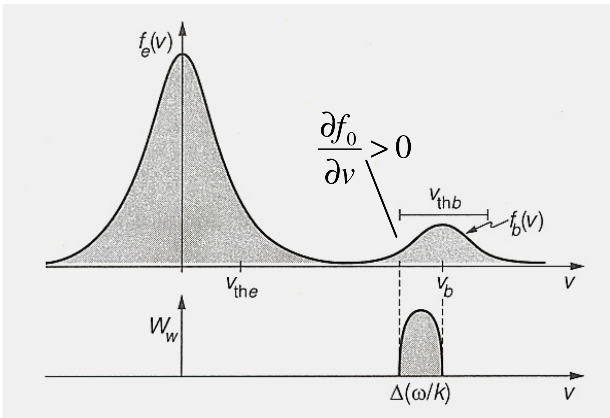
Physical explanation:

Particles are alternatively accelerated and decelerated in a wave, normally with little net effect. But when the phase velocity and the particle velocity are close, then the particles will be accelerated for wave phases with attractive E and decelerated for repulsive E so that a co-moving density modulation is the result. The wave will loose energy in accelerating slower particles ($v<v_{ph}$) and gain energy from the deceleration of particles with $v>v_{ph}$. For $\frac{\partial f_0}{\partial v} < 0$ there are more slower than faster particles and the wave will have a net loss of energy, but for $\frac{\partial f_0}{\partial v} > 0$ the kinetic energy of particles can be converted into wave energy. The particle distribution will become distorted by the formation of a plateau near the resonance as a result.



(Lesch)

Typical situations are those where an additional faster particle component (= beam) mixes with a plasma at rest. The distribution function rises again for the beam component, leading to the excitation of waves in the region of the positive gradient.



(Lesch)

A good example is the Bunemann electron-ion instability that is exited when a a fast electron beam is injected into a cold ion background.

In analogy to our example of the streaming instability above we may write the dispersion relation as

$$\frac{{\omega_{pi}}^2}{\omega^2} + \frac{{\omega_{pe}}^2}{(\omega - ku)^2} = 1$$

The characteristic frequency for a charge neutral plasma **n_i=n_e** is $\omega_{bun} = \left(\frac{m_e}{16 m_i}\right)^{\frac{1}{3}} \omega_{pe} \approx \frac{1}{30.9}\omega_{pe}$ for protons, with a growth rate of $\gamma_{bun} = \left(\frac{3 m_e}{16 m_i}\right)^{\frac{1}{3}} \omega_{pe} \approx \frac{1}{21}\omega_{pe}$. Again these are fast growing instabilities which lead to violent beam disruptions.

We have described and illustrated the conditions for plasma instabilities using particularly simple examples. There is a large variety of more complex velocity distribution functions where the instability condition is met in some interval leading to a host of different kind of instabilities,- as further discussed in the literature.

Final outlook:

A plasma is described by Maxwell's equations and the equation of motion. Observed quantities are ensemble averages of particlre velocity distribution functions. The set of coupled linear differential equations becomes a set of non-linear integro-differential equations. We have described small fluctuations by linear theory, but even simple examples show, that the constraints of linear theory are often violated after a few plasma periods. The full treatment of many astrophysical problems requires elaborate and expensive numerical tools such as particle-in-cell simulations and of course the inclusion of special relativity for the equation of motion and the propagation of field and potentials.

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